SOME TWO-DIMENSIONAL INCLUSION PROBLEMS IN ELASTICITY

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CERTIFICATE

This is to certify that the thesis entitled 'Some Two-Dimensional Inclusion Problems in Elasticity' that is being submitted by Shri O. P. Kapoor, M. A. for the award of the Degree of Doctor of Philosophy to the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standard fulfilling the requirements of the regulations to the Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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C O N T E N T S

I N T R O D U C T I O N

This thesis is concerned with the 'inclusion problems' in the infinitesimal theory of elasticity. Briefly the problem may be stated as follows :

A region (the 'inclusion') in an isotropic elastic medium undergoes a permanent change of form which, in the absence of the elastic constraints imposed by the surrounding material (the 'matrix'), would be a prescribed uniform strain. Find the elastic field in the matrix and inclusion. The inclusion may or may not have the same elastic properties as those of the matrix. If the properties are different, the inclusion will be termed as 'inhomogeneity'. Such dimensional changes in the inclusion could arise due to thermal effects, plastic flow or phase transformation, thereby generating a

system of locked-up accommodation stresses.

For a wide review and possible applications the article 'Elastic Inclusions and Inhomogeneities' by J. D. Eshelby in Progress in Solid Mechanics, Vol. II ((5)), may be referred to. However, it may be stated that the applications of such problems can be found to such diverse fields as engineering, physics and even atomic physics. The engineering applications of the theory of cavities may be read from 'Stress concentration around holes', (Pergamon Press, 1961) by G.N. Savin, or from 'Kerbspannungslehre', (Springer-Verlag, Berlin, 1958) by H. Neuber. As regards applications to physical problems, one finds its use in the theory of martensite transformations, the theory of cracks, etc. Recently much attention is being paid to determine the elastic constants C_{ijkl}, S_{ijkl}, of polycrystalline aggregates, where actual constants vary from grain to grain. The results have also been used in predictions of macroscopic elastic moduli of two phase composites. Hashin((8)), Hershey((9)), Hill((10)) - ((12)), Budiansky ((21)) and various other workers have tried to work out such problems.

The applications cited are essentially macroscopic. Elastic inclusions have also been found to be useful models of lattice defects in crystals. It is not at all obvious that a model of this type will be of any use in discussing the lattice defects and yet calculations are being made upon such hypothesis. Dundurs and Mura ((22)) have investigated the behaviour of an edge dislocation present in the matrix with a circular inclusion. Dundurs and Sendeckyj ((23)) have

studied a similar problem in which the dislocation is present inside the circular inclusion.

A simple problem of spherical inclusion in an infinite isotropic elastic medium was first studied by G. Frenkel ((7)), and then by Mott and Nabarro ((24)), and Nabarro ((25-26)), in connection with their theory of precipitation hardening in alloys. No progress could be made for next fifteen years or so. A systematic study for ellipsoidal inclusion was given by Eshelby ((4, 5)), where he made use of what may be described as 'point-force technique'. Three-dimensional inclusions which are more realistic are found to be intractable except in the simplest case. The two-dimensional problem is comparatively simpler to deal with, because of the application of complex variable method. A complex variable formulation to the inclusion problem was given by Bhargava and Jaswon ((6)). The problem, as far as it is related to elliptic inclusion of the same material as that of the matrix, was solved by them. If the elliptic inclusion material is different from that of the matrix (yet isotropic) the problem is still solvable, as was done by Bhargava and Radhakrishna ((13, 14)). Then a great stride was made by taking the inclusion and the matrix of different orthotropic materials ((15)). This was possible by what may be described as the application of energy principles to inclusion problems. Willis ((20)) has given the solution to the problem of an elliptic inclusion in a cubic material.

This thesis may be regarded as a continuation of such two-dimensional problems. It appears necessary to remark at this stage, that the previous

work has been upon elliptic inclusions in two-dimensions and ellipsoidal inclusions in three dimensions. It is not surprising, because of the fact that integrals involved are comparatively easily tractable in these cases. In the former case, the integrals are suitable Cauchy's integrals; in the latter the theory of ellipsoidal harmonics can be employed. For shapes other than elliptic, one has to be extremely careful, because of multivalued integrals which may be involved. In this thesis we have given the solutions to the rectangular and triangular inclusions.

In all the solutions so far, the matrix was supposed to be infinite in all directions. What happens if the matrix is semi-infinite, is yet another important class of problems, which has not been attempted so far, as far as is known to the author. It seems to have important applications in engineering, geophysics, metallurgy, etc. Damage to structures resulting from swelling of clay soils has been well documented over years. These problems have risen in connection with both undisturbed and compacted clays used as a foundation for structural frames and for slabs. In most instances, the damage has been attributed to vertical component of swelling and also the horizontal component. The swelling usually is due to the variation of moisture content of clays. A similar situation could arise where piles have been constructed in a soil mass. The simple model of homogeneous, isotropic elastic material which has been assumed in this thesis may be a simplification of the soil mass which is inelastic continuum. But this does yield a first approximation to the situations.

Another class of solutions refers to the interaction of inclusions and cavities in an otherwise infinite medium. It is hardly necessary to describe the importance of such problems in engineering structures, where the presence of a hole may considerably weaken the structure, because of extreme stress concentration, if the hole is too near the inclusion. In the atomic model the case where there may be a vacancy in the presence of misfit atoms may be discussed with the help of above model. Another class of problems refers to the interaction of the elastic fields of two inhomogeneities which are present near each other in the infinite medium. A few problems of these types have been included here. It appears to be the first attempt when three regions-two inhomogeneities and the matrix; or one hole, one inclusion and the matrix; or one inhomogeneity, one inclusion and the matrix are considered.

Before we begin to describe what has been done in each chapter, it seems necessary to state that the solution to the 'inclusion problem' by the point-force technique, hinges upon the evaluation of the elastic fields due to point-forces. This is known in simple two-dimensional and three-dimensional cases, and has been made use of at appropriate places, where we have taken the results from standard texts. However, if the holes or inhomogeneities are present in an infinite medium and a point-force is applied to the material, what is the elastic field everywhere ? These are problems of considerable importance in the theory of elasticity. We have solved a few problems and included them here.

First Chapter of this thesis explains the 'inclusion problem' and the 'point-force technique'. The original work starts from the second Chapter.

In Chapters II and III the problems of a rectangular and a triangular inclusion are solved. Ordinarily problems involving such regions are solved by means of conformal maps which in effect round off the corners and slightly deviate the sides from their straightness. But here we are able to dispense with such a technique and give exact analytical solutions which hold good even in neighbourhood of the corners.

Chapters IV and V deal with inclusion problems in semi-infinite medium. In Chapter IV a circular inclusion is taken up and in Chapter V a rectangular inclusion with a side parallel to the leading edge is considered. It is found that the edge effect is confined to a small region around the inclusions and when the distance of the inclusion is four to five times the radius of circular inclusion, or the length of the rectangular inclusion, the solutions differ slightly from those for the infinite case, the error being of the order of about two in hundred.

In Chapter VI the effect of a concentrated force which is applied at a point of the boundary of a circular inhomogeneity has been evaluated in terms of complex potential functions. In Chapter VII the case of a deforming inhomogeneity of rather a general type is considered. The results of Chapter VI have been utilised. Eshelby ((5)) has also suggested a method of solving such a problem. But his method was found to be quite cumbersome. The method developed here is direct and simpler. In Chapter VIII the complex functions yielding the effect of a concentrated force applied at any point of the elastic medium containing a cavity have been obtained and these results are applied in Chapter IX to solve the inclusion problem in the presence of a cavity. The results

could be applied in the theory of cracks which has played a part in Griffith's treatment of rupture (see for example, Sneddon ((29))).

In Chapter X the effect of a concentrated force acting at any point of a medium containing an inhomogeneity has been found out in terms of complex potentials. In Chapter XI the problem of an inclusion undergoing dimensional changes in the presence of a circular inhomogeneity has been dealt with. Some numerical work was done which has been included in the form of tables and figures at the end.

In the last Chapter the problem of two deforming inhomogeneities has been solved through an interesting process of superposition. This is the first time, when two inhomogeneities have been considered with elastic constants differing from the surroundings. Even though the solution is obtained in a surprisingly simple manner, the problem is the most important one in this thesis. Moreover the superposition technique used here can have very wide applications. It has lead us to feel strongly that the problem of simultaneous presence of any number of inclusions reasonably placed in the complex plane can be solved by the repeated application of the process indicated in the last Chapter.

The work presented in the Chapters II, III and VIII to XI is based on the following papers which have been published or are under publication.

1. Two-dimensional Rectangular inclusion.
   (Under publication in Proc. Nat. Inst. Sci. India.)

2. Two-dimensional trinngular inclusions in an elastic infinite medium.
(Bulletin de l' Academic Polonaise des Sciences, Vol. XI, No. 7 , 1963).

3. Circular inclusion in an infinite elastic medium with a circular hole.
(Proc. Camb. Phil. Soc. (1964), 60, 675).

4. Circular Inclusion in an infinite elastic medium with a circular inhomogeneity.
(Proc. Camb. Phil. Soc. (1966), 62, 113).

In addition, the contents of the last Chapter have been communicated to the Proc. Cambridge Philos. Soc. by Prof. J. G. Oldroyd and of Chapter IV to Proc. Nat. Inst. Sci. India , by Prof. R.S. Verma.

LIST OF SYMBOLS

$x, y; r, \theta$	Two-dimensional coordinate axes
$u, v; u_r, u_\theta$	Displacement components
$\tau_x, \tau_y, \tau_{xy}; \tau_r, \tau_\theta, \tau_{xy}$ --	Stress components
e_{xx}, e_{yy}, e_{xy}	Strain Components
ν	Poisson's ratio
λ, μ	Lame's constants
$\alpha = \frac{3-\nu}{1+\nu}$	for plane stress case
$\alpha = 3 - 4\nu$	for plane strain case
$i = \sqrt{-1}$	
Subscript i	denotes that subscripted quantity pertains to inclusion
Subscript ih	denotes that the subscripted quantity pertains to inhomogeneity
Subscript m	denotes that the subscripted quantity pertains to matrix
Bar	denotes the complex conjugate
Prime	denotes differentiation with respect to the argument

C H A P T E R I

INCLUSION PROBLEM AND POINT-FORCE

A large category of problems of the theory of elasticity which are important for practical applications and at the same time admit of considerable simplification in the mathematical aspect of the solution are two-dimensional problems. The mathematical formulation and the basic results are available in many books, e.g. ((1)), ((2)) and ((3)). The figures in the double parenthesis shall mean the reference numbers in the bibliography (page 196). The results which we shall need in this thesis are included below for ready reference as well as for familiarity with the notation.

The fundamental problem of plane elasticity in the absence of body forces is to solve the biharmonic equation

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4},$$

under appropriate boundary conditions. The Airy's function U is related to the stresses by the equations

$$\sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}.$$

Alternatively in terms of complex variable formalism, the main problem is to find two complex functions $\phi(z)$ and $\psi(z)$ of the complex variable $z \equiv x + iy$, satisfying appropriate boundary conditions. They are related to the stress and displacement components in the following way :

$$\left.\begin{aligned} \sigma_x + \sigma_y &= 4\,\mathrm{Re}\ \phi'(z) \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2\left[\ \bar{z}\,\phi''(z) + \psi'(z)\right] \end{aligned}\right\} \quad . \tag{1}$$

and

$$2\mu\,(u + iv) = \alpha\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)} \tag{2}$$

where $\alpha = 3 - 4\nu$ for plane strain case and $\alpha = \frac{3-\nu}{1+\nu}$ for generalised plane stress case.

Corresponding to a given state of stress of a body, the complex functions $\phi(z)$ and $\psi(z)$ change with a change in the coordinate system. If the origin of plane is shifted to a point z_0 without rotation of axes and if $\phi_1(z)$ and $\psi_1(z)$ are the functions in the new system, corresponding to $\phi(z)$ and $\psi(z)$ in the old system, it can be shown that

$$\phi_1(z) = \phi(z - z_0), \tag{3}$$

$$\psi_1(z) = \psi(z - z_0) - \bar{z}_0\,\phi'(z - z_0). \tag{4}$$

It is seen that the function $\psi(z)$ is not invariant for a translation of the axes i.e. , the values for the old functions cannot be derived by simply replacing z by $z - z_0$ in $\psi_1(z)$. In contrast, the function $\phi(z)$ is invariant. The effect of rotating the axes, leaving the origin fixed, and turningt the axes through an angle is given by the following equations

$$\phi_1(z) = \phi(z\,e^{-i\theta})\,e^{i\theta}, \tag{5}$$

$$\psi_1(z) = \psi(z\,e^{-i\theta})\,e^{-i\theta}. \tag{6}$$

Now we state the inclusion problem : A region (the inclusion) of a homogeneous, isotropic elastic body tends to undergo a change of form which in the absence of the surrounding material (the matrix) would be a prescribed homogeneous strain. Owing to the elastic constraints of the matrix locked up accommodation stresses develop in the system. Determination of the stresses and the equilibrium configuration is the inclusion problem. Let us use the term ' free-inclusion' for the free-surface configuration which could be attained by the inclusion, if the matrix were not there.

Unlike inclusions of simple shapes namely sphere or a circle, the equilibrium boundary of the general inclusion becomes an unknown of the

problem. Thus, for instance when the inclusion and the free-inclusion are similarly situated rectangles , the equilibrium boundary is not a similar rectangle, not even a rectangle (fig. 5, p. 135) . Generally speaking, to calculate the correct shape using direct procedures presents a difficult mathematical problem. However, very powerful indirect attack on such problems, based on the concept of the point-force, has been introduced by J.D. Eshelby ((4)). Even though his argument is a purely heuristic one, it has yielded very good results as has been illustrated in the works of Eshelby ((4, 5)), Jaswon and Bhargava ((6)), Bhargava and Sharma ((17, 18)) and Willis ((20)).

In brief the idea is explained int the following sequence of imaginary cutting, straining and welding operations.

Cut out the inclusion from the medium. Allow it to attain the free-state configuration. Now it can no longer be fitted without strain into the cavity from which it was taken. Apply surface tractions to the surface of the free-inclusion so as to reduce it to its original size. At this stage there will be a stress field present in the inclusion. We shall call this field 'the constrained- stress field' for future reference. Put it back in the cavity, maintaining the tractions. Weld the material together across the surface. At this stage no stresses appear in the matrix. A distribution of point-forces is now applied over the boundary, equal and opposite to the impressed surface tractions. In the absence of the matrix these point-forces , superposed on the tractions, would have the effect of restoring the inclusion to its free state dimensions. However, owing to the elastic

constraints of the matrix, this superposition produces an equilibrium configuration. In other words the effect of the deforming inclusion, so to say, is to bring into play a distribution of point-forces on its surface.

Thus, if we know by some means the stress-strain field associated with a single concentrated force applied at a point of the medium, the cumulative effect of the distribution of the point-forces on a surface can be obtained by the process of integration. The stress and strain fields in the matrix due to the inclusion will be the same as due to the point-forces distribution. In the inclusion, however, the stress field is obtained by superposing the stress field due to the forces, on the stress field already present there due to the impressed tractions.

Eshelby ((4)) solved the problem of an ellipsoidal inclusion by this method. The problem of a force acting at a point in an infinite elastic medium was first discussed by Lord Kelvin. A force (X,Y,Z), acting at (x,y,z) , produces a displacement (u , v , w) at (x_1, y_1, z_1) given by the formulae

$$u = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left[\frac{\lambda+3\mu}{\lambda+\mu}\frac{X}{d} + (x-x_1)\left\{\frac{X(x-x_1)+Y(y-y_1)+Z(z-z_1)}{d^3}\right\}\right],$$

where $d^2 = \sum (x-x_1)^2$; v, w have similar expressions.

For two-dimensional problems the expression for displacement at a point (x, y) due to a concentrated force acting at the point (x_1, y_1) is

$$2\mu u = -\frac{\alpha}{2\pi(1+\alpha)} X \log d = \frac{1}{2\pi(\alpha+1)}\left[X\,\frac{\{(x-x_1)^2-(y-y_1)^2\} + 2(x-x_1)(y-y_1)}{d^2}\right],$$

where $\alpha = 3-4\nu$ for plane strain and $\alpha = \frac{3-\nu}{1+\nu}$ for plane stress. There is a similar expression for v.

The complex functions $\phi(z)$ and $\psi(z)$, due to a concentrated force is given, say, in ((2, 3)). A point force $P = P_1 + iP_2$ acting at the point ζ of an unbounded body has the following derivatives of the complex functions $\phi(z)$ and $\psi(z)$ associated with it:

$$\phi'(z) = \frac{-P}{2\pi(\alpha+1)} \frac{1}{z-\zeta}, \tag{7}$$

$$\psi'(z) = \frac{\alpha \bar{P}}{2\pi(\alpha+1)} \frac{1}{z-\zeta} - \frac{\bar{\zeta} P}{2\pi(\alpha+1)} \frac{1}{(z-\zeta)^2}, \tag{8}$$

where $\bar{P}$ is the conjugate of P.

The cumulative effect of a distribution of point-forces acting along an arc γ of an infinite isotropic elastic medium can be obtained by integration of the individual effects of the concentrated forces. Thus, if $P(\zeta)$ is the distribution of forces given as a function of point ζ on γ, then

$$\phi'(z) = -\frac{1}{2\pi(\alpha+1)} \int_\gamma \frac{P\,ds}{z-\zeta}\,, \tag{9}$$

$$\psi'(z) = \frac{\alpha}{2\pi(\alpha+1)} \int_\gamma \frac{\bar{P}\,ds}{z-\zeta} - \int_\gamma \frac{\bar{\zeta} P\,ds}{2\pi(\alpha+1)(z-\zeta)^2}\,. \tag{10}$$

As an illustrative example we take a circular inclusion of unit radius which tends to expand to a size $1+\delta$, where δ is small so that the linear theory is applicable. In the presence of the matrix an intermediate size will be the equilibrium size. The point-force layer generated can be shown to be as given below :

$$P\,ds = (P_1 + iP_2)\,ds = -2i(\lambda+\mu)\,\delta\; d\zeta$$

$$\bar{P}\,ds = (P_1 - iP_2)\,ds = 2i(\lambda+\mu)\,\delta\; d\bar{\zeta}$$

The cumulative effect will be given by the integrals (9) and (10) where γ is now a unit circle. Evaluating the integrals we obtain

$$\phi'(z) = \frac{2(\lambda+\mu)\,\delta}{\alpha+1}\,, \tag{11}$$

$$\psi'(z) = 0, \tag{12}$$

for z in the inclusion, and

$$\phi'(z) = 0, \tag{13}$$

$$\psi'(z) = \frac{\alpha-1}{\alpha+1}(\lambda+\mu)\frac{2\delta}{z^2}, \tag{14}$$

for z in the matrix.

The stresses and displacements in the matrix can be found directly from (13) and (14) with the help of (1) and (2). But in the inclusion (11) and (12) give only a part of the stress field. The constrained-stress field given by

$$\sigma_x = -2(\lambda+\mu)\delta,$$
$$\sigma_y = -2(\lambda+\mu)\delta,$$

must be added to it to obtain the net stresses in the inclusion. Continuity of normal and tangential stresses across the boundary yields a check on the analysis.

CHAPTER II

RECTANGULAR INCLUSION

We now consider the following problem. A rectangular hole of sides 2a, 2b is cut out of an infinite isotropic elastic medium, and a rectangular inclusion of sides $2a(1 + \delta_1)$, $2b(1 + \delta_2)$ is forced symmetrically into the cavity. Here δ_1, δ_2 are small quantities within the limit of elastic strain. What will be the equilibrium size and shape of the inclusion, and the accompanying stress-strain field ?

Following the method proposed by Eshelby, we first reduce the inclusion from its free dimensions to the dimensions of the hole. This is effected by impressing upon it a displacement field

$$u^{\circ} = -\delta_1 x, \qquad v^{\circ} = -\delta_2 y \tag{15}$$

It can be seen that this transforms, to the first order, the rectangle

$$x = \pm 2a(1+\delta_1),$$

$$y = \pm 2b(1+\delta_2),$$

into the rectangle

$$x = \pm 2a,$$

$$y = \pm 2b.$$

Corresponding to (15) there exists a strain field

$$e^\circ_{xx} = -\delta_1, \qquad e^\circ_{yy} = -\delta_2, \quad e^\circ_{xy} = 0 \tag{16}$$

and a stress field

$$\left.\begin{aligned} \sigma_x^\circ &= -[\lambda(\delta_1+\delta_2) + 2\mu\delta_1], \\ \sigma_y^\circ &= -[\lambda(\delta_1+\delta_2) + 2\mu\delta_2], \\ \tau^\circ_{xy} &= 0. \end{aligned}\right\} \tag{17}$$

On the boundary the surface traction components

$$\begin{aligned} T_1^\circ &= \sigma_x^\circ \cos(x,n) + \tau^\circ_{xy} \cos(y,n), \\ T_2^\circ &= \tau^\circ_{xy} \cos(x,n) + \sigma_y^\circ \cos(y,n), \end{aligned} \tag{18}$$

are required per unit length, at a point where the outward normal to the boundary has direction cosines cos (x, n), cos (y, n). The opposing point-force components are

$$P_1 = -T_1^\circ \qquad \text{and} \qquad P_2 = -T_2^\circ. \tag{19}$$

We write the boundary conditions (18) in another fashion as follows

$$\begin{aligned} T_1^\circ &= \sigma_x^\circ \frac{dy}{ds} - \tau^\circ_{xy} \frac{dx}{ds}, \\ T_2^\circ &= \tau^\circ_{xy} \frac{dy}{ds} - \sigma_y^\circ \cdot \frac{dx}{ds}. \end{aligned} \tag{20}$$

Also, if $\zeta = x + iy$ signifies a boundary point,

$$\frac{d\zeta}{ds} = \frac{1}{2}\left(\frac{d\zeta}{ds} + \frac{d\bar\zeta}{ds}\right), \qquad \frac{d\bar\zeta}{ds} = \frac{dx}{ds} - \frac{i\,dy}{ds}.$$

Whence

$$\frac{dx}{ds} = \frac{1}{2}\left(\frac{d\zeta}{ds} + \frac{d\bar\zeta}{ds}\right), \qquad \frac{dy}{ds} = -\frac{i}{2}\left(\frac{d\zeta}{ds} - \frac{d\bar\zeta}{ds}\right). \tag{21}$$

Accordingly from (19), (20) and (21) ,

$$(P_1 + iP_2)\,ds = \frac{i}{2}\left\{(\sigma_x^\circ + \sigma_y^\circ)\,d\zeta - (\sigma_x^\circ - \sigma_y^\circ)\,d\bar\zeta\right\} + \tau_{xy}^\circ\,d\bar\zeta,$$

$$(P_1 - iP_2)\,ds = -\frac{i}{2}\left\{(\sigma_x^\circ + \sigma_y^\circ)\,d\bar\zeta - (\sigma_x^\circ - \sigma_y^\circ)\,d\zeta\right\} + \tau_{xy}^\circ\,d\zeta. \tag{22}$$

We have obtained equations (22) a little painstakingly because we shall consistently need them in the following chapters also. It may be noticed that the analysis from (18) to (22) does not depend upon the shape of the boundary. Thus the point-force distribution is given by

$$P\,ds = \frac{i}{2}\left\{(\sigma_x^\circ + \sigma_y^\circ)\,d\zeta - (\sigma_x^\circ - \sigma_y^\circ)\,d\bar\zeta\right\},$$

$$\bar P\,ds = -\frac{i}{2}\left\{(\sigma_x^\circ + \sigma_y^\circ)\,d\bar\zeta - (\sigma_x^\circ - \sigma_y^\circ)\,d\zeta\right\}.$$

Substituting for $P\,ds$ and $\bar P\,ds$ in (9) and (10), the complex functions are as follows

$$\phi'(z) = \frac{1}{2\pi(\alpha+1)}\left[-\frac{i}{2}(\sigma_x^\circ + \sigma_y^\circ)\int_\gamma \frac{d\zeta}{z-\zeta} + \frac{i}{2}(\sigma_x^\circ - \sigma_y^\circ)\int_\gamma \frac{d\bar\zeta}{z-\zeta}\right],$$

$$\psi'(z) = \frac{1}{2\pi(\alpha+1)}\left[\frac{i\alpha}{2}(\sigma_x^\circ - \sigma_y^\circ)\int_\gamma \frac{d\zeta}{z-\zeta} - \frac{i\alpha(\sigma_x^\circ - \sigma_y^\circ)}{2}\int_\gamma \frac{d\bar\zeta}{z-\zeta}\right.$$

$$\left. - \frac{i}{2}(\sigma_x^\circ + \sigma_y^\circ)\int_\gamma \frac{\bar\zeta\,d\zeta}{(z-\zeta)^2} + \frac{i}{2}(\sigma_x^\circ - \sigma_y^\circ)\int_\gamma \frac{\bar\zeta\,d\bar\zeta}{(z-\zeta)^2}\right],$$

where γ is now the rectangle. The integrals can be evaluated by splitting each integral into four integrals, each in turn along one side of the rectangle. Care has to be taken to evaluate the integrals, as they involve multivalued functions. The resulting values of the functions $\phi'(z)$ and $\psi'(z)$ are

$$\phi'(z) = \frac{-(\sigma_x^\circ + \sigma_y^\circ)}{4\pi(\alpha+1)} (\theta_1 + \theta_2 + \theta_3 + \theta_4) + \frac{\sigma_x^\circ - \sigma_y^\circ}{4\pi(\alpha+1)} \Big[(\theta_1 - \theta_2 + \theta_3 - \theta_4) + i \log \frac{(x_1^2+y_1^2)(x_2^2+y_2^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)} \Big], \tag{23}$$

$$\psi'(z) = \frac{\sigma_x^\circ - \sigma_y^\circ}{4\pi} \frac{(\alpha-1)}{(\alpha+1)} (\theta_1 + \theta_2 + \theta_3 + \theta_4) - \frac{\sigma_x^\circ + \sigma_y^\circ}{4\pi}\left(\frac{\alpha-1}{\alpha+1}\right) \times (\theta_1 - \theta_2 + \theta_3 - \theta_4) - \frac{i(\sigma_x^\circ + \sigma_y^\circ)}{4\pi}\left(\frac{\alpha-1}{\alpha+1}\right) \times \log \frac{(x_1^2+y_1^2)(x_2^2+y_2^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)} + i\,\frac{\sigma_x^\circ - \sigma_y^\circ}{\pi}\,\frac{ab}{\alpha+1} \Big[\frac{2xy_2 - i(a^2+y_2^2-x^2)}{(x_2^2+y_2^2)(x_1^2+y_2^2)} + \frac{2xy_1 - i(a^2-x^2+y_1^2)}{(x_2^2+y_1^2)(x_1^2+y_1^2)} - \frac{2yx_1 + i(b^2-y^2+x_1^2)}{(x_1^2+y_1^2)(x_1^2+y_2^2)} - \frac{2yx_2 + i(b^2-y^2+x_2^2)}{(x_2^2+y_1^2)(x_2^2+y_2^2)} \Big], \tag{24}$$

where $x_1 = x+a$, $x_2 = x-a$, $y_1 = y+b$ and $y_2 = y-b$.

The angles $\theta_1, \theta_2, \theta_3, \theta_4$ are angles subtended by sides AB, BC, CD and DA respectively at the point z. An angle will be positive or negative according as it is traced anti-clockwise or clockwise. This has to ascertained from the Fig. 1 p. 132. Obviously

$$\begin{aligned} \theta_1 + \theta_2 \pm \theta_3 + \theta_4 &= 0, \quad \text{for } z \text{ in the matrix,} \\ &= 2\pi, \quad \text{for } z \text{ in the inclusion.} \end{aligned}$$

Differentiating (23) with respect to z we obtain

$$\phi''(z) = \frac{\sigma_x^\circ - \sigma_y^\circ}{2\pi(\alpha+1)} \Bigg[\frac{2axy_2 - ia(a^2 - x^2 + y_2^2)}{(x_1^2+y_2^2)(x_2^2+y_2^2)} - \frac{b(b^2-y^2+x_1^2) - 2ibx_1y}{(x_1^2+y_1^2)(x_1^2+y_2^2)}$$
$$+ \frac{ia(a^2 - x^2 + y_1^2) - 2axy_1}{(x_1^2+y_1^2)(x_2^2+y_1^2)}$$
$$- \frac{2ibx_2y - b(b^2 - y^2 + x_2^2)}{(x_2^2+y_1^2)(x_2^2+y_2^2)} \Bigg]. \qquad (25)$$

The stresses in the matrix are directly given by substituting the values of $\phi'(z)$, $\phi''(z)$ and $\psi'(z)$ in (1). However for inclusion we must add to the stress field the constrained-stress field given by (17). Hence the stress field in the inclusion is given by

$$(\sigma_x + \sigma_y)_i = \frac{\alpha-1}{\alpha+1}(\sigma_x^\circ + \sigma_y^\circ) + \frac{\sigma_x^\circ - \sigma_y^\circ}{\pi(\alpha+1)}(\theta_1 + \theta_3 - \theta_2 - \theta_4)$$

$$(\sigma_y - \sigma_x + 2i\tau_{xy})_i = \frac{-2(\sigma_x^\circ - \sigma_y^\circ)}{\alpha+1} - \frac{(\sigma_x^\circ + \sigma_y^\circ)(\alpha-1)}{2\pi(\alpha+1)}(\theta_1 - \theta_2 + \theta_3 - \theta_4)$$

$$- \frac{i(\sigma_x^\circ + \sigma_y^\circ)}{2\pi} \quad \frac{\alpha - 1}{\alpha + 1} \quad \log \frac{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}{(x_1^2 + y_2^2)(x_2^2 + y_1^2)}$$

$$- \frac{\sigma_x^\circ - \sigma_y^\circ}{2\pi(\alpha+1)} \left[(\bar{z} + 2ib) \left(\frac{-2axy_2 + ia(a^2 - x^2 + y_2^2)}{(x_1^2 + y_2^2)(x_2^2 + y_2^2)} \right) \right.$$

$$+ (\bar{z} - 2a) \left(\frac{b(b^2 - y^2 + x_1^2) - 2byx_1}{(x_1^2 + y_1^2)(x_1^2 + y_2^2)} \right) + (\bar{z} - 2ib) \left(\frac{2axy_1 - ia(a^2 - x^2 + y_1^2)}{(x_1^2 + y_1^2)(x_2^2 + y_1^2)} \right)$$

$$\left. + (\bar{z} + 2a) \left(\frac{-b(b^2 - y^2 + x_2^2) + 2ibx_2}{(x_2^2 + y_1^2)(x_2^2 + y_2^2)} \right) \right].$$

It can be readily checked that the normal and tangential stress components are continuous across the boundary, proving thus, that the analysis is correct. On the other hand hoop stress is discontinuous which it should be on physical grounds. The jump in the hoop stress as we cross the boundary is

$$\frac{2(\lambda + \mu)(\delta_1 + \delta_2)(\alpha - 1)}{\alpha + 1} + \frac{4\mu(\delta_1 - \delta_2)}{\alpha + 1}$$

An important particular case can be derived when $\delta_1 = \delta_2$ which means that the initial stress field in the inclusion is uniform. In the matrix the stress field is

$$(\sigma_x + \sigma_y)_m = 0,$$

$$(\sigma_y - \sigma_x + 2i\tau_{xy})_m = -\frac{\sigma_x^\circ}{\pi} \frac{\alpha - 1}{\alpha + 1} \Big[\theta_1 - \theta_2 + \theta_3 - \theta_4$$

$$+ i \log \frac{(x_2^2 + y_2^2)(x_1^2 + y_1^2)}{(x_1^2 + y_2^2)(x_2^2 + y_1^2)} \Big]. \qquad (26)$$

It is at once seen that the stress field in the matrix is free from

dilatation and is only of distortion. It is of interest to observe the variation of the stresses σ_x and σ_y on the boundary. In Fig. 2, 3, p. 132, 133, we have drawn the graphs of $\sigma_x / \sigma_x^{\circ}$ and $\tau_{xy} / \sigma_x^{\circ}$ for different values of a/b, taking Poisson's ratio $\nu = 1/3$ in plane stress case.

As regards the stresses in the inclusion (for $\delta_1 = \delta_2$), the stress field is given

$$(\sigma_x + \sigma_y)_i = 2\sigma_x^{\circ} \frac{\alpha - 1}{\alpha + 1},$$

$$(\sigma_y - \sigma_x + 2i\tau_{xy})_i = -\frac{\sigma_x^{\circ}}{\pi} \frac{\alpha - 1}{\alpha + 1}\left[(\theta_1 - \theta_2 + \theta_3 - \theta_4) + i \log \frac{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}{(x_1^2 + y_2^2)(x_2^2 + y_1^2)} \right].$$

The maximum shearing stress in the matrix is given by

$$\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right]_m^{1/2}$$

Numerical evaluation of the expression can easily be made from (26). Lines of constant shearing stress have been drawn for plane stress case taking $\nu = 1/3$ for different values of a/b in Fig. 4, p. 134. Clearly the shearing stress is highly concentrated near the corners and in fact it tends to infinity at the corners. It may be explained by observing that at points very near the corners, the shearing stress is sufficiently large to produce plastic deformations and that linear theory cannot be applied to the region very near the corners. It can also be seen that shearing stresses are dominant on

the narrow sides of the rectangle. Similar results are seen in the case of an elliptic inclusion. When $a/b = 10$, the lines of shearing stress are substantially the same as in the case of an ellipse with axial ratio equal to 10. The later were drawn for the case an elliptic inclusion from ((6)).

As regards the displacement field, we integrate (23) and (24) with respect to z and obtain

$$\phi(z) = -\frac{\sigma_x^0+\sigma_y^0}{4\pi(\alpha+1)}(\theta_1+\theta_2+\theta_3+\theta_4)z - \frac{\sigma_x^0-\sigma_y^0}{4\pi(\alpha-1)}$$

$$\times\Bigg[y\log\frac{(x_2^2+y_2^2)(x_1^2+y_1^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)} + b\log\frac{(x_1^2+y_2^2)(x_1^2+y_1^2)}{(x_2^2+y_2^2)(x_2^2+y_1^2)}$$

$$-x(\theta_1-\theta_2+\theta_3-\theta_4)+2a(\theta_2-\theta_4)$$

$$+i\Bigg\{x\log\frac{(x_1^2+y_2^2)(x_2^2+y_1^2)}{(x_1^2+y_1^2)(x_2^2+y_2^2)} + a\log\frac{(x_2^2+y_2^2)(x_1^2+y_2^2)}{(x_1^2+y_1^2)(x_2^2+y_1^2)}$$

$$-y(\theta_1-\theta_2+\theta_3-\theta_4)+2b(\theta_1-\theta_3)\Bigg\}\Bigg],$$

$$\psi(z) = \frac{\sigma_x^0-\sigma_y^0}{4\pi}\;\frac{\alpha-1}{\alpha+1}(\theta_1-\theta_2+\theta_3-\theta_4)z$$

$$\frac{\sigma_x^0+\sigma_y^0}{4\pi}\;\frac{\alpha-1}{\alpha+1}\Bigg[y\log\frac{(x_2^2+y_2^2)(x_1^2+y_1^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)}$$

$$+b\log\frac{(x_1^2+y_2^2)(x_1^2+y_1^2)}{(x_2^2+y_2^2)(x_2^2+y_1^2)} - x(\theta_1-\theta_2+\theta_3-\theta_4)$$

$$+ 2a(\theta_2-\theta_4) + i\left\{x \log \frac{(x_1^2+y_2^2)(x_2^2+y_1^2)}{(x_1^2+y_1^2)(x_2^2+y_2^2)}\right.$$

$$\left.+ a \log \frac{(x_2^2+y_2^2)(x_1^2+y_2^2)}{(x_1^2+y_1^2)(x_2^2+y_1^2)} - y(\theta_1-\theta_2+\theta_3-\theta_4)+2b(\theta_1-\theta_3)\right\}\Bigg]$$

$$+ \frac{\sigma_x^\circ - \sigma_y^\circ}{4\pi(\alpha+1)}\left[b \log \frac{(x_1^2+y_1^2)(x_1^2+y_2^2)}{(x_2^2+y_2^2)(x_2^2+y_1^2)} - 2a(\theta_2-\theta_4)\right.$$

$$\left. + i\left\{ a \log \frac{(x_1^2+y_1^2)(x_2^2+y_1^2)}{(x_1^2+y_2^2)(x_2^2+y_2^2)} + 2b(\theta_1-\theta_3)\right\}\right].$$

The displacements are obtained by substituting the above functions along with $\phi'(z)$ given in (23) in (2). In particular the displacements of the internal boundary of the matrix can be evaluated. Regarding the inclusion, it had two displacements- one when it is reduced to the size of the whole and second when it deforms due to the application of the layer of point-forces. It can be easily shown that the later is continuous, and has the same value at the boundary , as the whole in the matrix. Numerical values for the displacements in the case $\delta_1 = \delta_2 = \delta$ were obtained in terms of δ . In Fig. 5,p 135 , we have drawn schematically the equilibrium shapes of inclusions for which a/b = 1 , 2, and 10, for the case of plane stress taking ν = 1/3.

CHAPTER III

TRIANGULAR INCLUSION

In this chapter the problem of a triangular inclusion is solved. The method adopted is essentially the same as in the case of a rectangular inclusion in the previous chapter.

Choosing the coordinate system suitably, the corners of the constrained inclusion triangle may be labelled as A(0,0), B(a,0) and C(b, c), Fig. 6, p. 136 . The homogeneous deformation which the inclusion would undergo if it were free , may be prescribed as

$$u = \delta_1 \left(x - \frac{a+b}{3}\right) + \delta_3 \left(y - \frac{c}{3}\right),$$
$$v = \delta_3 \left(x - \frac{a+b}{3}\right) + \delta_2 \left(y - \frac{c}{3}\right), \qquad (27)$$

where δ_1, δ_2 and δ_3 are of the order of permissible strains in linear theory. If the above deformation is opposed, a constant

stress field given by

$$\left.\begin{aligned} \sigma_x^0 &= -\{\lambda(\delta_1+\delta_2)+2\mu\delta_1\}, \\ \sigma_y^0 &= -\{\lambda(\delta_1+\delta_2)+2\mu\delta_2\}, \\ \tau_{xy}^0 &= -2\mu\delta_3, \end{aligned}\right\} \tag{28}$$

is developed in the inclusion. The point-force layer for this may be obtained on similar lines as the equation (22)

$$\begin{aligned} P\,ds &= \tfrac{1}{2}i\{(\sigma_x^0+\sigma_y^0)\,d\zeta + (\sigma_y^0-\sigma_x^0+2i\tau_{xy}^0)\,d\bar{\zeta}\}, \\ \bar{P}\,ds &= -\tfrac{1}{2}i\{(\sigma_x^0+\sigma_y^0)\,d\bar{\zeta} + (\sigma_y^0-\sigma_x^0-2i\tau_{xy}^0)\,d\zeta\}. \end{aligned} \tag{29}$$

Now γ is the contour of the triangle. Substituting (29) in equations (9, 10) we shall have the complex functions in the form of integrals. They are evaluated by splitting each integral into three integrals, each one along one side of the boundary. If the radius vector joining a fixed point (either in the matrix or in the inclusion) to a moving point on the boundary, traces an angle in the anticlockwise direction, the angle will be positive, but negative otherwise. Thus if $\theta_1, \theta_2, \theta_3$ are angles subtended by sides AB, BC, and CA respectively, at any point z, Fig. 6 p. 136 , the complex functions are

$$\begin{aligned} \phi'(z) = &-\frac{\sigma_x^0+\sigma_y^0}{4\pi(\alpha+1)}(\theta_1+\theta_2+\theta_3) + i\left\{\frac{\sigma_y^0-\sigma_x^0+2i\tau_{xy}^0}{4\pi(\alpha+1)}\right\} \\ &\times\Big[\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2}+i\theta_1 \\ &+\frac{b-a-ic}{b-a+ic}\Big(\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2}+i\theta_2\Big) \\ &+\frac{b-ic}{b+ic}\Big(\frac{1}{2}\log\frac{x^2+y^2}{x_1^2+y_1^2}+i\theta_3\Big)\Big], \end{aligned} \tag{30}$$

where $\quad x_1 = x-a, \quad x_2 = x-b, \quad y_1 = y-c.$

$$\psi'(z) = \frac{-\alpha(\sigma_y^\circ - \sigma_x^\circ - 2i\tau_{xy}^\circ)}{4\pi(\alpha+1)}(\theta_1 + \theta_2 + \theta_3)$$

$$+ \frac{i(\sigma_x^\circ + \sigma_y^\circ)}{4\pi}\,\frac{\alpha-1}{\alpha+1}\left[\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2} + i\theta_1\right.$$

$$+ \frac{b-a-ic}{b-a+ic}\left(\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2} + i\theta_2\right)$$

$$\left. + \frac{b-ic}{b+ic}\left(\frac{1}{2}\log\frac{x^2+y^2}{x_1^2+y_1^2} + i\theta_3\right)\right]$$

$$- \frac{i(\sigma_x^\circ+\sigma_y^\circ)}{4\pi(\alpha+1)}\left[z\left(\frac{a(x^2-y^2-ax)+iy(a^2-2ax)}{(x^2+y^2)(x_1^2+y^2)}\right)\right.$$

$$+\left(\frac{2iac}{b-a+ic} + \frac{b-a-ic}{b-a+ic}z\right)\left\{\frac{-axx_1+bxx_2}{}\right.$$

$$\frac{+ay^2-c^2x_1+2cyx_1-a^2b+ab^2+i(yb^2}{}$$

$$\left.\frac{-ya^2+yc^2-2bxy+2axy-cy^2+cx^2+ca^2-2acx)}{(x_2^2+y_1^2)(x_1^2+y^2)}\right\}$$

$$+\frac{b-ic}{b+ic}z\left\{\frac{-bxx_2+c^2x-2cxy+by^2}{}\right.$$

$$\left.\frac{+i(cyy_1-yb^2+2bxy-cx^2)}{(x_2^2+y_1^2)(x^2+y^2)}\right\} -$$

$$-\frac{i(\sigma_y^0-\sigma_x^0+2i\tau_{xy}^0)}{4\pi(\alpha+1)}\left[\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2}+i\theta_1\right.$$

$$+\left(\frac{b-a-ic}{b-a+ic}\right)^2\left\{\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2}+i\theta_2\right\}$$

$$+\left(\frac{b-ic}{b+ic}\right)^2\left\{\frac{1}{2}\log\frac{x^2+y^2}{x_1^2+y_1^2}+i\theta_3\right\}$$

$$+Z\left\{\frac{a(xx_1-y^2)+iy(a^2-2ax)}{(x^2+y^2)(x_1^2+y^2)}\right\}$$

$$+\left\{\frac{2iac(b-a-ic)}{(b-a+ic)^2}+\left(\frac{b-a-ic}{b-a+ic}\right)^2 Z\right\}$$

$$\times\left(\frac{-axx_1+bxx_2+ay^2-c^2x_1+2cyx_1-a^2b+ab^2+i(yb^2-ya^2+yc^2-2bxy+2axy-cy^2+cx^2+ca^2-2acx)}{(x_2^2+y_1^2)(x_1^2+y^2)}\right)$$

$$\left.+\left(\frac{b-ic}{b+ic}\right)^2 Z\left(\frac{-bxx_2+c^2x-2cxy+by^2+i(cyy_1-yb^2+2bxy-cx^2)}{(x_2^2+y_1^2)(x^2+y^2)}\right)\right]. \quad (31)$$

It may be remembered that

$$\theta_1 + \theta_2 + \theta_3 = 2\pi, \text{ if } z \text{ lies inside the inclusion,}$$
$$= 0, \text{ if } z \text{ lies in the matrix.}$$

Differentiating (30) with respect to z, we get

$$\begin{aligned}
\Phi''(z) = {} & \frac{i(\sigma_y^\circ - \sigma_x^\circ + 2i\tau_{xy}^\circ)}{4\pi(\alpha+1)} \left[\frac{a(xx_1 - y^2) + iy(a^2 - 2ax)}{(x_1^2+y^2)(x^2+y^2)} \right. \\
& + \frac{b-a-ic}{b-a+ic} \left(\frac{\begin{array}{c} -axx_1 + bxx_2 + ay^2 - c^2x_1 + 2cx_1y \\ -a^2b + ab^2 + i(yb^2 + yc^2 + 2axy - cy^2 \\ -ya^2 + \cancel{yc^2} - 2bxy \cancel{- cy^2} + cx^2 - cx^2 - 2axc) \end{array}}{(x_2^2+y_1^2)^2(x_1^2+y^2)} \right) \\
& \left. + \frac{b-ic}{b+ic} \left(\frac{\begin{array}{c} -bxx_2 + c^2x - 2cxy + by^2 \\ + i(cy^2 - yb^2 - yc^2 + 2bxy - cx^2) \end{array}}{(x_2^2+y_1^2)(x^2+y^2)} \right) \right].
\end{aligned} \tag{32}$$

The stress field in the matrix is obtained directly by substituting (30), (31) and (32) in (1). However, for the stresses in the inclusion we have to add the initial stresses (28) to those given by the complex function and (1). It can be easily verified that the normal and tangential stresses are continuous in each side of the triangle.

The interesting case of pure dilatation ($\delta_1 = \delta_2$, $\delta_3 = 0$) is studied further. In this case the stress field in the inclusion is given by

$$(\sigma_x + \sigma_y)_i = (\sigma_x^0 + \sigma_y^0) \frac{\alpha - 1}{\alpha + 1},$$

$$(\sigma_y - \sigma_x + 2i\tau_{xy})_i = \frac{\sigma_x^0+\sigma_y^0}{4\pi}\;\frac{\alpha-1}{\alpha+1}\left[\frac{c(b-a)}{(b-a)^2+c^2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2}\right.$$

$$+\frac{bc}{b^2+c^2}\log\frac{x^2+y^2}{x_2^2+y_1^2} - \theta_1 + \frac{(b-a)^2-c^2}{(b-a)^2+c^2}\theta_2$$

$$+\frac{b^2-c^2}{b^2+c^2}\theta_3 + i\left\{\log\frac{x_1^2+y^2}{x^2+y^2}\right.$$

$$+\frac{(b-a)^2-c^2}{(b-a)^2+c^2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2} + \frac{b^2-c^2}{b^2+c^2}\log\frac{x_2^2+y_1}{x^2+y^2}$$

$$\left.\left.+\frac{4c(b-a)}{(b-a)^2+c^2}\theta_2 + \frac{4bc}{b^2+c^2}\theta_3\right\}\right].$$

Lines of maximum shearing stress in the inclusion have been drawn in Fig.7, p. 137 , for a few particular cases. The maximum shearing stress is highly concentrated near the corners as expected.

Integrating (30) and (31) with respect to z

$$\phi(z) = -\frac{\sigma_x^0+\sigma_y^0}{4\pi(\alpha+1)}(\theta_1+\theta_2+\theta_3)z$$

$$-\,i\,\frac{(\sigma_y^0-\sigma_x^0+2i\tau_{xy})}{4\pi(\alpha+1)}\left[\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2}+i\theta_1\right.$$

$$+\left(\frac{2iac}{b-a+ic}+\frac{b-a-ic}{b-a+ic}z\right)\left(\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2}+i\theta_2\right)$$

$$\left.+\frac{b-ic}{b+ic}z\left(\frac{1}{2}\log\frac{x^2+y^2}{x_2^2+y_1^2}+i\theta_3\right)\right]. \qquad (33)$$

$$\Psi(z) = \frac{\sigma_y^\circ - \sigma_x^\circ + 2i\,\tau_{xy}^\circ}{4\pi} \quad \frac{\alpha}{\alpha+1} \left(\theta_1 + \theta_2 + \theta_3\right) z$$

$$+ \frac{i(\sigma_x^\circ + \sigma_y^\circ)}{4\pi} \quad \frac{\alpha-1}{\alpha+1} \Bigg[z\left(\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2} + i\theta_1\right)$$

$$+ \left(\frac{2iac}{b-a+ic} + \left(\frac{b-a-ic}{b-a+ic}\right)z\right)\left(\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2} + i\theta_2\right)$$

$$+ \frac{b-ic}{b+ic}\, z\left(\frac{1}{2}\log\frac{x^2+y^2}{x_2^2+y_1^2} + i\theta_3\right)\Bigg]$$

$$+ \frac{i(\sigma_y^\circ - \sigma_x^\circ + 2i\,\tau_{xy}^\circ)}{4\pi(\alpha+1)} \Bigg[z\left(\frac{1}{2}\log\frac{x_1^2+y^2}{x^2+y^2} + i\theta_1\right)$$

$$+ \left(\frac{2iac(b-a-ic)}{(b-a+ic)^2} + \left(\frac{b-a-ic}{b-a+ic}\right)^2 z\right) \qquad (34)$$

$$\times \left(\frac{1}{2}\log\frac{x_2^2+y_1^2}{x_1^2+y^2} + i\theta_2\right)$$

$$+ \left(\frac{b-ic}{b+ic}\right)^2 z\left(\frac{1}{2}\log\frac{x^2+y^2}{x_2^2+y_1^2} + i\theta_3\right)\Bigg].$$

The displacement fields in the inclusion and the matrix can be obtained through the complex potentials and the equation (2). For the particular case of pure dilation equilibrium shapes for a few cases have been drawn in Fig. 8 , p. 137 .

CHAPTER IV

CIRCULAR INCLUSION IN A HALF PLANE

An isolated force P is acting at the point ζ of an elastic isotropic medium which is supposed occupying the upper half of the complex plane. Green and Zerna ((3)) have given the complex functions $\phi(z)$ and $\psi(z)$ which give the stress-displacement fields due to the point-force P . They were modified to a form suited to us and are given below .

$$\phi'(z) = \frac{-P}{2\pi(\alpha+1)}\left(\frac{1}{z-\zeta} + \frac{\alpha}{z-\bar{\zeta}}\right) + \frac{P}{2\pi(\alpha+1)}\left(\frac{1}{z-\bar{\zeta}} + \frac{\zeta-z}{(z-\bar{\zeta})^2}\right), \tag{35}$$

$$\psi'(z) = \frac{\bar{P}}{2\pi(\alpha+1)}\left\{\frac{\alpha}{z-\zeta} + \frac{3z-\zeta}{(z-\bar{\zeta})^2} - \frac{2z(z-\zeta)}{(z-\bar{\zeta})^3}\right\}$$

$$+\frac{P}{2\pi(\alpha+1)}\left\{\frac{-\bar{\zeta}}{(z-\zeta)^2} + \frac{\alpha}{z-\bar{\zeta}} - \frac{\alpha z}{(z-\bar{\zeta})^2}\right\}, \qquad (36)$$

where $\bar{P}$ is complex conjugate of P ; $\alpha = (3-\nu)/(1+\nu)$ for plane stress case and $\alpha = 3-4\nu$ for plane strain case. Thus, when there is a continuous layer of point-forces acting along an arc γ of the half plane, the cumulative effect will be given by

$$\phi'(z) = \frac{1}{2\pi(\alpha+1)}\left\{\int_\gamma \frac{-P\,ds}{z-\zeta} - \int_\gamma \frac{\alpha P\,ds}{z-\bar{\zeta}}\right.$$

$$\left. + \int_\gamma \frac{\bar{P}\,ds}{z-\bar{\zeta}} - \int \frac{\bar{P}(z-\zeta)\,ds}{(z-\bar{\zeta})^2}\right\}, \qquad (37)$$

$$\psi'(z) = \left\{\int_\gamma \frac{\alpha\bar{P}\,ds}{z-\zeta} + \int_\gamma \frac{\bar{P}(3z-\zeta)\,ds}{(z-\bar{\zeta})^2} - \int_\gamma \frac{2\bar{P}z(z-\zeta)}{(z-\zeta)^3}\right.$$

$$\left. - \int_\gamma \frac{P\bar{\zeta}\,ds}{(z-\zeta)^2} + \int_\gamma \frac{\alpha P\,ds}{z-\bar{\zeta}} - \int_\gamma \frac{z\alpha P\,ds}{(z-\bar{\zeta})^2}\right\}. \qquad (38)$$

Let us now consider the case of a circular inclusion of radius unity and its centre at a distance ℓ from the leading edge. $(z-i\ell)(z+i\ell)\leq 1$ can be taken to represent the inclusion, y - axis

passing through the centre as shown in the Fig. 9, p. 138 . Complex potentials, stresses and displacements obtained due to a layer of point-forces will be marked with subscripts i and m according as they refer to the inclusion or to the matrix.

The inclusion in the absence of the matrix tends to undergo the displacements characterised by

$$\begin{aligned} u &= \delta_1 x + \delta_3 (y - \ell), \\ v &= \delta_2 (y - \ell) + \delta_3 x, \end{aligned} \tag{39}$$

whence the homogeneous strains are

$$e_{xx} = \delta_1, \qquad e_{xy} = \delta_3, \qquad \text{and} \qquad e_{yy} = \delta_2 .$$

First we shall consider the case of principal strains ($\delta_3 = 0$). In the later part of this chapter, the case of pure shear will be dealt with. If the deformations (39) are opposed, the stress field generated in the inclusion will be

$$\left.\begin{aligned} \sigma_x^0 &= -\{\lambda(\delta_1+\delta_2) + 2\mu\delta_1\}, \\ \sigma_y^0 &= -\{\lambda(\delta_1+\delta_2) + 2\mu\delta_2\}, \\ \tau_{xy} &= -2\mu\delta_3 = 0 \end{aligned}\right\} \tag{40}$$

The point-force distribution which comes into play on the boundary $(z - i\ell)(\bar{z} + i\ell) = 1$ in this case as found from (40) and (32), will be

$$\left.\begin{aligned} P\,ds &= -i(\lambda+\mu)(\delta_1+\delta_2)\,d\zeta + i\mu(\delta_1-\delta_2)\,d\bar{\zeta}, \\ \bar{P}\,ds &= -i\mu(\delta_1-\delta_2)\,d\zeta + i(\mu+\lambda)(\delta_1+\delta_2)\,d\bar{\zeta}. \end{aligned}\right\} \tag{41}$$

We substitute these expressions in (35) and (36) and evaluate the

contour integrals. It may be noted that γ is the inclusion boundary where we have the following relations

$$\bar{\zeta} = \frac{1}{\zeta - i\ell} - i\ell \quad \text{and} \quad d\bar{\zeta} = \frac{-1}{(\zeta - i\ell)^2}\, d\zeta$$

These relations are given because they are used in the integration. The expressions look simpler, if the substitutions $z_1 = z + i\ell = r_1 e^{i\theta_1}$, $z_2 = z - i\ell = r_2 e^{i\theta_2}$, are made. Thus,

$$\begin{aligned}\Phi_i'(z) = {} & \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} - \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\,\frac{1}{z_1^2} \\ & + \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(-\frac{1}{z_1^2} + \frac{4i\ell}{z_1^3} + \frac{3}{z_1^4}\right),\end{aligned} \tag{42}$$

$$\begin{aligned}\Psi_i'(z) = {} & \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left(\frac{-1}{z_1^2} + \frac{2i\ell}{z_1^3}\right) \\ & - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(\alpha - \frac{10i\ell}{z_1^3} - \frac{12\ell^2}{z_1^4} - \frac{9}{z_1^4} + \frac{12i\ell}{z_1^5}\right),\end{aligned} \tag{43}$$

$$\begin{aligned}\Phi_m'(z) = {} & -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\,\frac{1}{z_1^2} \\ & + \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(-\frac{1}{z_1^2} - \frac{1}{z_2^2} + \frac{4i\ell}{z_1^3} + \frac{3}{z_1^4}\right),\end{aligned} \tag{44}$$

$$\begin{aligned}\Psi_m'(z) = {} & \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left(\frac{1}{z_2^2} - \frac{1}{z_1^2} + \frac{2i\ell}{z_1^3}\right) \\ & - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(\frac{3}{z_2^4} - \frac{2i\ell}{z_2^3} - \frac{10i\ell}{z_1^3} \right. \\ & \left. - \frac{12\ell^2}{z_1^4} + \frac{12i\ell}{z_1^5} - \frac{9}{z_1^4}\right).\end{aligned} \tag{45}$$

The stress field may be found from the above potentials by means of (1). But it must be emphasized that the inclusion had an initial stress field given by (40) and this must be added to the one got from the functions $\phi_i'(z)$ and $\psi_i'(z)$. Before proceeding further we can verify that the normal and tangential stresses are continuous across the inclusion boundary, of course, on the edge $y = 0$ the normal and tangential stresses vanish, as they should.

Stresses in the cartesian form are given below :

$$
\begin{aligned}
(\sigma_x)_i = & -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{1+(x^2-y_1^2)\frac{1}{r_1^2} \\
& + 4l(3y_1x^2-y_1^3)\frac{1}{r_1^6} + 2(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^6}\Big\} \\
& - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{1+2(x^2-y_1^2)\frac{1}{r_1^4} + 4l(3y_1x^2-y_1^3)\frac{1}{r_1^6} \\
& + (2r_1^2+24l^2+3)(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^8} \\
& - 12(r_1^2+2)(5y_1x^4+y_1^5-10x^2y_1^3)\frac{1}{r_1^{10}} \\
& - 12(x^6-15x^4y_1^2+15x^2y_1^4-y_1^6)\frac{1}{r_1^{10}}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\sigma_y)_i = & -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{-1+(x^2-y_1^2)\frac{3}{r_1^4} \\
& -(3y_1x^2-y_1^3)\frac{4\ell}{r_1^6}-2(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^6}\Big\} \\
& +\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{1-2(x^2-y_1^2)\frac{1}{r_1^4} \\
& +\frac{20\ell}{r_1^6}(3y_1x^2-y_1^3) \\
& +(2r_1^2+15+24\ell^2)(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^8} \\
& -12\ell(r_1^2+2)(5y_1x^4-10x^2y_1^3+y_1^5)\frac{1}{r_1^{10}} \\
& -12(x^6-15x^4y_1^2+15x^2y_1^4-y^6)\frac{1}{r_1^{10}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(\tau_{xy})_i = & \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\,x\Big\{\frac{2y_1}{r_1^4} \\
& +\frac{4\ell}{r_1^6}(x^2-3y_1^2)+\frac{8}{r_1^6}(y_1^3-x^2y_1)\Big\} \\
& +\frac{4\mu(\delta_1-\delta_2)}{\alpha+1}\,x\Big\{\frac{3\ell}{r_1^6}(x^2-3y_1^2)+\frac{6}{r_1^{10}}(3y_1x^4+10y_1^3x^2+3y_1^5) \\
& +(2r_1^2+12\ell^2+9)(y_1^3-x^2y_1)\frac{1}{r_1^8} \\
& -\frac{3\ell}{r_1^{10}}(r^2+2)(x^4-10x^2y_1^2+5y_1^4)\Big\},
\end{aligned}
$$

where

$$y_1=y+\ell,\quad y_2=y-\ell,\quad r_1^2=x^2+y_1^2,\quad r_2^2=x^2+y_2^2 \quad\text{and}\quad r^2=x^2+y^2 .$$

The stresses in the matrix are directly obtained by the potentials $\phi_m'(z)$ and $\psi_m'(z)$. They are

$$(\sigma_x)_m = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left\{\frac{x^2-y_1^2}{r_1^2}+\frac{x^2-y_2^2}{r_2^2}\right.$$

$$+\frac{4l}{r_1^6}(3y_1x^2-y_1^3)+\frac{2}{r_1^6}\left.(x^4-6x^2y_1^2+y_1^4)\right\}$$

$$-\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left\{\frac{2}{r_2^4}(x^2-y_2^2)+\frac{2}{r_1^4}(x^2-y_1^2)\right.$$

$$+\frac{4l}{r_1^6}(3y_1x^2-y_1^3)+(2r_1^2+3+24l^2)(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^8}$$

$$+\frac{(2r_2^2-3)}{r_2^8}(x^4-6x^2y_2^2+y_2^4)$$

$$-\frac{12l}{r_1^{10}}(r_2^2+2)(5y_1x^4-10x^2y_1^3+y_1^5)$$

$$-\frac{12}{r_1^{10}}\left.(x^6-15y_1^2x^4+15x^2y_1^4-y_1^6)\right\},$$

$$(\sigma_y)_m = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left\{(x^2-y_2^2)\frac{1}{r_2^4}-\frac{3}{r_1^4}(x^2-y_1^2)\right.$$

$$+\frac{4l}{r_1^6}(3y_1^2x-y_1^3)+\frac{2}{r_1^6}\left.(x^4-6x^2y_1^2+y_1^4)\right\}$$

$$+\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left\{-\frac{2}{r_1^4}(x^2-y_1^2)-\frac{2}{r_2^2}(x^2-y_2^2)\right.$$

$$+\frac{20l}{r_1^6}(3y_1x^2-y_1^3)+(2r_1^2+15+24l^2)(x^4-6x^2y_1^2+y_1^4)\frac{1}{r_1^8}$$

$$+(2r_2^2+3)(x^4-6x^2y_2^2+y_2^4)\frac{1}{r_2^8}$$

$$-\frac{12l}{r_1^{10}}(r_1^2+2)(5y_1x^4-10x^2y_1^3+y_1^5)$$

$$-\frac{12}{r_1^{10}}\left.(x^6-15x^4y_1^2+15x^2y_1^4-y_1^6)\right\},$$

$$(\tau_{xy})_m = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left\{\frac{2y_1}{r_1^4} - \frac{2y_2}{r_2^4} + \frac{4\ell}{r_1^6}(x^2-3y_1^2)\right.$$

$$\left. + \frac{8}{r_1^6}(y_1^3 - y_1 x^2)\right\} + \frac{4\mu(\delta_1-\delta_2)}{\alpha+1}\left\{(2r_2^2-3)(y_2^3 - x^3 y_2)\frac{1}{r_2^8}\right.$$

$$- \frac{12\ell}{r_1^{10}}(r_1^2+2)(x^4 - 10x^2y_1^2 + 5y_1^4) + \frac{1}{r_1^8}(y_1^3 - x^2 y_1)$$

$$\left.\times (2r_1^2 + 24\ell^2 + 9) + \frac{3\ell}{r_1^6}(x^2-3y_1^2) + \frac{6}{r_1^{10}}(3y_1x^4 - 10x^2y_1^3 + 3y_1^5)\right\}.$$

The hoop stress σ_x on the edge can be found from the expression for $(\sigma_x)_m$ given above by putting y = 0 .

$$(\sigma_x)_m\Big|_{y=0} = \frac{-4(\delta_1+\delta_2)(\lambda+\mu)(\alpha-1)}{\alpha+1}\,\frac{x^2-\ell^2}{r_1^4}$$

$$+ \frac{4\mu(\delta_1-\delta_2)}{\alpha+1}\left\{-\frac{2}{r_1^4}(x^2-\ell^2)\right.$$

$$\left. + \frac{4\ell^2}{r_1^6}(3x^2-\ell^2) + \frac{3}{r_1^8}(x^4 - 6x^2\ell^2 + \ell^4)\right\}.$$

The normal and tangential stresses continuously transmitted by the bond on the boundary of the inclusion are given below :

$$(\sigma_n)_{r_2=1} = -\frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left\{1 + \frac{2}{r_1^2}\cos 2\theta_1\right.$$

$$+ \frac{2r}{r_1^3}\cos(2\theta_2 - 3\theta_1 - \theta) - \frac{1}{r_1^2}\cos(2\theta_2 - 2\theta_1)$$

$$\left. - \frac{2\ell}{r_1^3}\sin(2\theta_2 - 3\theta_1)\right\} - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left\{\cos 2\theta_2\right.$$

$$+ \frac{2}{r_1^2}\cos 2\theta_1 - \frac{8\ell}{r_1^3}\sin 3\theta_1 - \frac{6}{r_1^4}\cos 4\theta_1$$

$$+ \frac{2r}{r_1^3}\cos(2\theta_2-3\theta_1-\theta) + \frac{12\ell r}{r_1^4}\sin(2\theta_2-4\theta_1-\theta) - \frac{10\ell}{r_1^3}\sin(2\theta_2-3\theta_1)$$
$$+ \frac{3(3+4\ell^2)}{r_1^4}\cos(2\theta_2-4\theta_1) + \frac{12\ell}{r_1^5}\sin(2\theta_2-5\theta_1)\Big\}$$

$$(\tau_{ns})\Big|_{r_2=1} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{\frac{2r}{r_1^3}\sin(2\theta_2-3\theta_1-\theta) - \frac{1}{r_1^2}\sin(2\theta_2-2\theta_1)$$
$$+ \frac{2\ell}{r_1^3}\cos(2\theta_2-3\theta_1)\Big\} + \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{\sin 2\theta_2$$
$$+ \frac{2r}{r_1^3}\sin(2\theta_2-3\theta_1-\theta) - \frac{12\ell r}{r_1^4}\cos(2\theta_2-4\theta_1-\theta)$$
$$- \frac{12r}{r_1^5}\sin(2\theta_2-5\theta_1-\theta) + \frac{10\ell}{r_1^3}\cos(2\theta_2-3\theta_1)$$
$$+ \frac{3(3+4\ell^2)}{r_1^4}\sin(2\theta_2-4\theta_1) - \frac{12\ell}{r_1^5}\cos(2\theta_2-5\theta_1)\Big\}.$$

The hoop stress is discontinuous across the boundary:

$$(\sigma_s)_i\Big|_{r_2=1} = \frac{-(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{1 + \frac{2}{r_1^2}\cos 2\theta_1 + \frac{1}{r_1^2}\cos(2\theta_2-2\theta_1)$$
$$- \frac{2r}{r_1^3}\cos(2\theta_2-3\theta_1-\theta) + \frac{2\ell}{r_1^3}\sin(2\theta_2-3\theta_1)\Big\}$$
$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{\cos 2\theta_2 - \frac{2}{r_1^2}\cos 2\theta_1 + \frac{8\ell}{r_1^3}\sin 3\theta_1$$
$$+ \frac{6}{r_1^4}\cos 4\theta_1 + \frac{2r}{r_1^3}\cos(2\theta_2-3\theta_1-\theta) + \frac{12\ell r}{r_1^4}\sin(2\theta_2-4\theta_1-\theta)$$
$$- \frac{10\ell}{r_1^3}\sin(2\theta_2-3\theta_1) + \frac{3}{r_1^4}(3+4\ell^2)\cos(2\theta_2-4\theta_1)$$
$$- \frac{12r}{r_1^5}\cos(2\theta_2-5\theta_1-\theta) + \frac{12\ell}{r_1^5}\sin(2\theta_2-5\theta_1)\Big\},$$

$$(\sigma_s)_m\Big|_{r_2=1} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{1-\frac{2}{r_1^2}\cos 2\theta_1+\frac{1}{r_1^2}\cos(2\theta_2-2\theta_1)$$

$$+\frac{2r}{r_1^3}\cos(2\theta_2-3\theta_1-\theta)-\frac{2\ell}{r_1^3}\sin(2\theta_2-3\theta_1)\Big\}$$

$$+\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{-3\cos 2\theta_2-\frac{2}{r_1^2}\cos 2\theta_1+\frac{8\ell}{r_1^3}\sin 3\theta_1$$

$$+\frac{6}{r_1^4}\cos 4\theta_1+\frac{2r}{r_1^3}\cos(2\theta_2-3\theta_1-\theta)-10\frac{\ell}{r_1^4}\sin(2\theta_2-3\theta_1)$$

$$+\frac{12\ell r}{r_1^4}\sin(2\theta_2-4\theta_1-\theta)-\frac{12r}{r_1^5}\cos(2\theta_2-5\theta_1-\theta)$$

$$+\frac{12\ell}{r_1^5}\sin(2\theta_2-5\theta_1)+\frac{3(3+4\ell^2)}{r_1^4}\cos(2\theta_2-4\theta_1)\Big\}.$$

In Table 1, we have given the values of normal, tangential and hoop stresses along the boundary of the inclusion for various values of distance ℓ. In Figs.16-23, p. 141-144, these have been graphically shown for the case when $\ell = 1.5$.

The stresses between the points A and B, Fig. 9 p. 138, will be of greater interest from practical point of view. The stress field near the inclusion and the leading edge was evaluated, for plane stress case taking Poisson's ratio equal to 1/3. Fig.11-12 p. 139 shows the variation of stresses at B with the variation of the parameter ℓ. It is observed that, when this parameter is more than 5, the stresses in the matrix would differ from those in a similar region of an infinite plate, with an inclusion embedded in it, by about five percent. And therefore, for all practical purposes, for ℓ greater

than five, the model may be taken to be a circular inclusion in an infinite medium. It is also seen that when the inclusion is almost touching the edge, the tangential stress on the inclusion boundary is maximum at the points D and E, Fig. 9 p. 138 , where θ_2 almost equals -65° and 245° . More over the hoop stress σ_s at the point B is greater in the case when $\delta_1 = \delta$, $\delta_2 = 0$, than that in the case when $\delta_1 = 0$, $\delta_2 = \delta$. In Fig. 13 p. 140 the variation of hoop stress σ_x at the point A of the leading edge has been shown.

The case of pure shear can be dealt with in a similar fashion. In this case we have in equation (39) $\delta_1 = 0$, $\delta_2 = 0$ and $\delta_3 \neq 0$. The relevant complex potential functions are given below.

$$\phi_i'(z) = \frac{2\mu\delta_3}{\alpha+1}\left(\frac{i}{z_1^2} + \frac{4l}{z_1^3} - \frac{3i}{z_1^4}\right), \tag{46}$$

$$\psi_i'(z) = \frac{2\mu\delta_3}{\alpha+1}\left(i\alpha + \frac{10l}{z_1^3} - \frac{12il^2}{z_1^4} - \frac{9i}{z^4} - \frac{12l}{z_1^5}\right), \tag{47}$$

$$\phi_m'(z) = \frac{2\mu\delta_3}{\alpha+1}\left(\frac{i}{z_1^2} - \frac{i}{z_2^2} + \frac{4l}{z_1^3} - \frac{3i}{z_1^4}\right), \tag{48}$$

$$\begin{aligned}\psi_m'(z) = \frac{2\mu\delta_3}{\alpha+1}\Bigg(&-\frac{3i}{z_2^4} - \frac{2l}{z_2^3} + \frac{10l}{z_1^3} \\ &\cancel{\frac{2\mu\delta_3}{\alpha+1}}\ \cancel{\frac{i}{z_2^4}} - \cancel{\frac{2l}{z_2^3}} - \cancel{\frac{10l}{z_1^3}} \\ &-\frac{9i}{z_1^4} - \frac{12il^2}{z_1^4} - \frac{12l}{z_1^5}\Bigg)\Bigg\}.\end{aligned} \tag{49}$$

Once the displacement fields in the matrix and in the inclusion are known, we may determine the equilibrium shape of the inclusion from one of them. Of course, when we talk of displacement field in the inclusion we mean the displacement field measured relatively to the stage at which the inclusion has been reduced to the dimensions of the hole.

In the case of principal strains, integrating the expressions (42), (43), (44) and (45) of p. 28 , we obtain

$$\phi_i(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} z_2 + \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\frac{1}{z_1}$$
$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(\frac{1}{z_1} - \frac{2il}{z_1^2} - \frac{1}{z_1^3}\right), \tag{50}$$

$$\psi_i(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left(\frac{1}{z_1} - \frac{il}{z_1^2}\right)$$
$$+ \frac{il(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}$$
$$\times \left(\alpha z_2 + \frac{5il}{z_1^2} + \frac{4l^2}{z_1^3} + \frac{3}{z_1^3} - \frac{3il}{z_1^4}\right), \tag{51}$$

$$\phi_m(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\frac{1}{z_1}$$
$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left(\frac{1}{z_1} + \frac{1}{z_2}\right.$$
$$\left. - \frac{2il}{z_1^2} - \frac{1}{z_1^3}\right), \tag{52}$$

$$\psi_m'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left(\frac{1}{z_1} - \frac{1}{z_2} - \frac{il}{z_1^2}\right)$$

$$- \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left\{\frac{5il}{z_1^2} + \frac{4l^2}{z_1^3} + \frac{3}{z_1^3}\right.$$

$$\left. - \frac{3il}{z_1^4} + \frac{il}{z_2^2} - \frac{1}{z_2^3}\right\}. \tag{53}$$

The displacement fields both in the inclusion and the matrix are given directly by the potential functions and the equation (2). In cartesian form

$$(2\mu u)_i = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1} z\left(1 - \frac{1}{r_1^2}\right.$$

$$\left. + \frac{\alpha}{r_1^2} + \frac{4ly_1}{r_1^4} + (x^2-3y_1^2)\frac{1}{r_1^4}\right)$$

$$+ \frac{\alpha\mu x(\delta_1-\delta_2)}{\alpha+1}\left\{1 + \frac{1}{r_1^2} - \frac{4ly_1}{r_1^4}\right.$$

$$\left. - (x^2-3y_1^2)\frac{1}{r_1^6}\right\}$$

$$+ \frac{\mu(\delta_1-\delta_2)x}{\alpha+1}\left\{\frac{12ly_1}{r_1^4} + (r_1^2+3+8l^2)\right.$$

$$x(x^2-3y_1^2)\frac{1}{r_1^6} - \frac{8l}{r_1^8}(3+2r_1^2)(y_1x^2-y_1^3)$$

$$\left. - \frac{3}{r_1^8}(x^4 - 10x^2y_1^2 + 5y_1^4)\right\},$$

$$(2\mu v)_i = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1} \left\{ y_2 - \frac{y_1}{r_1^2} - \frac{\alpha y_1}{r_1^2} \right.$$

$$\left. - \frac{2\ell}{r_1^4}(x^2-y_1^2) + \frac{1}{r_1^4}(3y_1x^2 - y_1^3) \right\}$$

$$+ \frac{\alpha\mu(\delta_1-\delta_2)}{\alpha+1} \left\{ y_2 + \frac{y_1}{r_1^2} + \frac{2\ell}{r_1^4}(x^2-y_1^2) \right.$$

$$\left. - \frac{3}{r_1^6}(3y_1x^2 - y_1^3) \right\} + \frac{\mu(\delta_1-\delta_2)}{\alpha+1} \left\{ \frac{-6\ell}{r_1^4} \right.$$

$$+ \frac{1}{r_1^6}(3y_1x^2 - y_1^3)(r_1^2 + 3 + 8\ell^2)$$

$$\left. - \frac{8\ell}{r_1^8}(3+2r_1^2)(y_1x^3 - y_1^3x) - \frac{3}{r_1^8}(5y_1x^4 - 10x^2y_1^3 + y_1^5) \right\},$$

$$(2\mu u)_m = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\, x \left\{ \frac{\alpha}{r_1^2} - \frac{1}{r_1^2} + \frac{1}{r_2^2} \right.$$

$$\left. + \frac{4\ell y_1}{r_1^4} + (x^2-3y_1^2)\frac{1}{r_1^4} \right\} + \frac{\alpha\mu(\delta_1-\delta_2)}{\alpha+1}\, x$$

$$\times \left\{ \frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{4\ell y_1}{r_1^4} - \frac{x^2-3y_1^2}{r_1^6} \right\}$$

$$+ \frac{\mu(\delta_1-\delta_2)x}{\alpha+1} \left\{ \frac{12\ell y_1}{r_1^4} + \frac{1}{r_1^6}(x^2-3y_1^2)(r_1^2+3+8\ell^2) \right.$$

$$+ \frac{1}{r_2^6}(r_2^2-1)(x^2-3y_2^2) - \frac{8\ell}{r_1^8}(3+2r_1^2)$$

$$\left. \times (y_1x^2 - y_1^3) - \frac{3}{r_1^8}(x^4 - 10x^2y_1^2 + 5y_1^4) \right\},$$

$$(2\mu v)_m = \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\left\{\frac{-\alpha y_1}{r_1^2} - \frac{y_1}{r_1^2} + \frac{y_2}{r_2^2}\right.$$

$$\left. - \frac{2l}{r_1^4}(x^2-y_1^2) + (3y_1x^2-y_1^3)\frac{1}{r_1^4}\right\} + \frac{\alpha\mu(\delta_1-\delta_2)}{\alpha+1}$$

$$\times\left\{\frac{-y_1}{r_1^2} - \frac{y_2}{r_2^2} - \frac{2l}{r_1^4}(x^2-y_1^2) + (3y_1x^2-y_1^3)\frac{1}{r_1^6}\right\}$$

$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\left\{-\frac{6l}{r_1^4}(x^2-y_1^2) + \frac{1}{r_1^6}(3y_1x^2-y_1^3)\right.$$

$$\times(r_1^2+3+8l^2) + \frac{(r_2^2-1)}{r_2^6}(3y_2x^2-y_2^3) - \frac{8l}{r_1^8}(3+2r_1^2)$$

$$\left.\times(y_1x^3-y_1^3x) - \frac{3}{r_1^8}(5y_1x^4-10x^2y_1^3+y_1^5)\right\}.$$

In the case of pure shear only the complex functions required for evaluating the displacement fields from (2), have been listed below.

$$\phi_i(z) = \frac{2\mu\delta_3}{\alpha+1}\left(\frac{-i}{z_1} - \frac{2l}{z_1^2} + \frac{i}{z_1^3}\right),$$

$$\psi_i(z) = \frac{2\mu\delta_3}{\alpha+1}\left(i\alpha z_2 - \frac{5l}{z_1^2} + 4i\frac{l^2}{z_1^3} + \frac{3i}{z_1^3} + \frac{3l}{z_1^4}\right),$$

$$\phi_m(z) = \frac{2\mu\delta_3}{\alpha+1}\left(\frac{-i}{z_1} + \frac{i}{z_2} - \frac{2l}{z_1^2} + \frac{i}{z_1^3}\right),$$

$$\psi_m(z) = \frac{2\mu\delta_3}{\alpha+1}\left(\frac{-5l}{z_1^2} + \frac{4il^2}{z_1^3} + \frac{3i}{z_1^3}\right.$$

$$\left. + \frac{3l}{z_1^4} + \frac{l}{z_2^2} + \frac{i}{z_2^3}\right).$$

CHAPTER V

RECTANGULAR INCLUSION IN A SEMI-INFINITE ELASTIC MEDIUM

In the analysis of this chapter we have evaluated the stress fields for a rectangular inclusion in a semi-infinite medium. The size of the rectangule is 2a x 2b . Its centre is at a distance c from the leading edge. The coordinate axes and the configuration are made clear in Fig. 24 , p. 145 . The inclusion tends to undergo the following deformation in the absence of the matrix.

$$u = \delta_1 x + \delta_3 (y-c),$$

$$v = \delta_3 x + \delta_2 (y-c).$$

If this deformation is opposed, the following stresses develop in the inclusion:

$$\left.\begin{aligned} \sigma_x^\circ &= -\{\lambda(\delta_1+\delta_2)+2\mu\delta_1\}, \\ \sigma_y^\circ &= -\{\lambda(\delta_1+\delta_2)+2\mu\delta_2\}, \\ \tau_{xy}^\circ &= -2\mu\delta_3 . \end{aligned}\right\} \tag{54}$$

It will be convenient to deal with principal strains and pure shear cases separately. For principal strain we shall assume that $\delta_3 = 0$. The case of shear will be taken up later on.

The point-force layer developed can be calculated from (54) and (32) . Thus

$$\begin{aligned} P\,ds &= -i(\lambda+\mu)(\delta_1+\delta_2)\,d\zeta + i\mu(\delta_1-\delta_2)\,d\bar{\zeta}, \\ \bar{P}\,ds &= i(\lambda+\mu)(\delta_1+\delta_2)\,d\bar{\zeta} - i\mu(\delta_1-\delta_2)\,d\zeta . \end{aligned}$$

substituting these expressions in (35) and (36) and taking γ to be the rectangular boundary of the inclusion, one can obtain the complex potential functions for the principal strains case. The integrals can be evaluated by a similar process as in the case of rectangular inclusion of chapter V. Each integral is split in four integrals one each along a side of the rectangle. In this way the integrals reduce to ordinary Riemann integrals. Care of course has to be taken since multivalued functions are involved. The results of integration and further simplification are that

$$\phi'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\pi(\alpha+1)} \Big\{ \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4)$$

$$+ \frac{1-\alpha}{2}(\theta_1' - \theta_2' + \theta_3' - \theta_4')$$

$$+ \frac{i(\alpha-1)}{2} \log \frac{(x_2^2+y_1^2)(x_1^2+y_2^2)}{(x_1^2+y_1^2)(x_2^2+y_2^2)} \Big\}$$

$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1} \Big\{ \frac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \theta_1' - \theta_2' + \theta_3' - \theta_4')$$

$$+ \frac{2x_1(c+b)}{x_1^2+y_1^2} + \frac{2x_2(c-b)}{x_2^2+y_2^2} - \frac{2x_1(c-b)}{x_1^2+y_2^2}$$

$$- \frac{2x_2(c+b)}{x_2^2+y_1^2} + i\Big\{ \frac{1}{2} \log \frac{(x_1^2+y_3^2)(x_2^2+y_4^2)}{(x_2^2+y_3^2)(x_1^2+y_4^2)}$$

$$+ \frac{1}{2} \log \frac{(x_1^2+y_1^2)(x_2^2+y_2^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)}$$

$$- \frac{x_2^2+y_1y_3}{x_2^2+y_1^2} + \frac{x_1^2+y_1y_3}{x_1^2+y_1^2}$$

$$+ \frac{x_2^2+y_2y_4}{x_2^2+y_2^2} - \frac{x_1^2+y_2y_4}{x_1^2+y_2^2} \Big\},$$

(55)

$$\psi'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} \Big[\Big\{ \frac{1-\alpha}{2}(\theta_1 - \theta_2 + \theta_3$$

$$- \theta_4 - \theta_1' + \theta_2' - \theta_3' + \theta_4')$$

$$+ \frac{x_2y_1 - xy_3 + y_1x - x_1y - \alpha(yx_1 - xy_1)}{x_1^2+y_1^2}$$

$$+\frac{-x_2y_2+x_1y_4+(1+\alpha)(yx_1-xy_2)}{x_1^2+y_2^2}$$

$$+\frac{x_1y_2-x_2y_4-(1+\alpha)(yx_2-xy_2)}{x_2^2+y_2^2}$$

$$+\frac{-x_1y_1+x_2y_3+(1+\alpha)(yx_2-xy_1)}{x_2^2+y_1^2}$$

$$+i\Bigg\{\frac{\alpha-1}{2}\log\frac{(x_2^2+y_3^2)(x_1^2+y_4^2)(x_2^2+y_2^2)(x_1^2+y_1^2)}{(x_1^2+y_3^2)(x_2^2+y_1^2)(x_1^2+y_2^2)(x_2^2+y_4^2)}$$

$$-\frac{x_1x_2+y_1y_3+(1+\alpha)(xx_2+yy_1)}{x_2^2+y_1^2}$$

$$+\frac{x_1x_2+y_2y_4+(1+\alpha)(xx_2+yy_2)}{x_2^2+y_2^2}$$

$$+\frac{x_1x_2+y_1y_3+(1+\alpha)(xx_1+yy_1)}{x_1^2+y_1^2}$$

$$-\frac{xx_2+y_2y_4+(1+\alpha)(xx_1+yy_2)}{x_1^2+y_2^2}\Bigg\}\Bigg]$$

$$+\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Bigg[\frac{1-\alpha}{2}(\theta_1+\theta_2+\theta_3+\theta_4)$$

$$+\frac{y_1(2x+x_2)-x_2(2y+y_3)}{x_2^2+y_1^2}+\frac{y_2(2x+x_1)-x_1(2y+y_4)}{x_1^2+y_2^2}$$

$$-\frac{y_2(2x+x_2)-x_2(2y+y_4)}{x_2^2+y_2^2}-\frac{y_1(2x+x_1)-x_1(2y+y_3)}{x_1^2+y_1^2}$$

$$+\frac{2x_2y_2(xx_2-yy_4)-(x_2^2-y_2^2)(x_2y+xy_4)}{(x_2^2+y_2^2)^2}$$

$$-\frac{2x_2y_1(xx_2-yy_3)-(x_2^2-y_1^2)(x_2y+xy_3)}{(x_2^2+y_1^2)^2}$$

$$+\frac{2x_1y_1(xx_1-yy_3)-(x_1^2-y_1^2)(x_1y+xy_3)}{(x_1^2+y_1^2)^2}$$

$$-\frac{2x_1y_2(x_1x-yy_4)-(x_1^2-y_2^2)(x_1y+xy_4)}{(x_1^2+y_2^2)^2}+\frac{ay_3+x_2(c+b)}{x_2^2+y_3^2}$$

$$-\frac{ay_4+x_2(c-b)}{x_2^2+y_4^2}+\frac{ay_3-x_1(c+b)}{x_1^2+y_3^2}-\frac{ay_4-x_1(c-b)}{y_4^2+x_1^2}$$

$$+i\Bigg\{\frac{x_2(2x+x_2)+y_1(2y+y_3)}{x_2^2+y_1^2}-\frac{x_2(2x+x_2)+y_2(2y+y_4)}{x_2^2+y_2^2}$$

$$+\frac{x_1(2x+x_1)+y_2(2y+y_4)}{x_1^2+y_2^2}-\frac{x_1(2x+x_1)+y_1(2y+y_3)}{x_1^2+y_1^2}$$

$$+\frac{1}{2}\log\frac{(x_1^2+y_3^2)(x_2^2+y_4^2)}{(x_2^2+y_3^2)(x_1^2+y_4^2)}+\frac{(x_2^2-y_2^2)(xx_2-yy_4)+2x_2y_2(x_2y+xy_4)}{(x_2^2+y_2^2)^2}$$

$$-\frac{(x_2^2-y_1^2)(x_2x-yy_3)+2x_2y_1(x_2y+xy_3)}{(x_2^2+y_1^2)^2}$$

$$+\frac{(x_1^2-y_1^2)(x_1x-yy_3)+2x_1y_1(yx_1+xy_3)}{(x_1^2+y_1^2)^2}$$

$$-\frac{(x_1^2-y_2^2)(x_1x-yy_4)+2x_1y_2(x_1y+xy_4)}{(x_1^2+y_2^2)^2}$$

$$+\frac{ax_2-y_3(c+b)}{x_2^2+y_3^2}-\frac{ax_2-y_4(c-b)}{x_2^2+y_4^2}$$

$$+\frac{ax_1+y_3(c+b)}{x_1^2+y_3^2}-\frac{ax_1+y_4(c-b)}{x_1^2+y_4^2}\Bigg\}\Bigg]. \qquad (56)$$

where the following abbreviations have been used.

$$x_1 = x + a, \qquad x_2 = x - a,$$

$$y_1 = y + c + b, \qquad y_2 = y + c - b,$$

$$y_3 = y - c - b, \quad \text{and} \quad y_4 = y - c + b.$$

The angles $\theta_1, \theta_2, \theta_3$ and θ_4 are angles subtended by the sides AB, BC, CD and DA respectively at the point z. An angle will be positive or negative according as it is traced anti-clockwise or clockwise. The angles $\theta_1', \theta_2', \theta_3'$ and θ_4' are angles subtended by the sides A'B', B'C', C'D' and D'A' of the image rectangle Fig. 24 ,p. 145 . To determine the sign one has to again see whether it is traced anti-clockwise or clockwise.

Differentiating (55) with respect to z we obtain

$$\phi''(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1}\Bigg[\ (\alpha+3)\left\{\frac{-y_2}{x_2^2+y_2^2} + \frac{y_1}{x_2^2+y_1^2}\right.$$

$$\left. - \frac{y_1}{x_1^2+y_1^2} + \frac{y_2}{x_1^2+y_2^2}\right\} + i\,(\alpha+3)\left\{\frac{x_2}{x_2^2+y_1^2}\right.$$

$$\left. - \frac{x_2}{x_2^2+y_2^2} - \frac{x_1}{x_1^2+y_1^2} + \frac{x_1}{x_1^2+y_2^2}\right\}\Bigg]$$

$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Bigg[\ \frac{-2y_1}{x_2^2+y_1^2} + \frac{2y_2}{x_2^2+y_2^2}$$

$$- \frac{2y_2}{x_1^2+y_2^2} + \frac{2y_1}{x_1^2+y_1^2} + \frac{y_4}{x_2^2+y_4^2}$$

$$- \frac{y_3}{x_2^2+y_3^2} + \frac{y_3}{x_1^2+y_3^2} - \frac{y_4}{x_1^2+y_4^2}$$

$$
\begin{aligned}
&+ \frac{y_4 \,(x_2^2 - y_2^2) - 2x_2^2\, y_2}{(x_2^2 + y_2^2)^2} \\
&+ \frac{-y_3 \,(x_2^2 - y_1^2) + 2x_2^2\, y_1}{(x_2^2 + y_1^2)^2} \\
&+ \frac{y_3 \,(x_1^2 - y_1^2) - 2x_1^2\, y_1}{(x_1^2 + y_1^2)^2} \\
&+ \frac{-y_4 \,(x_1^2 - y_2^2) + 2x_1^2\, y_2}{(x_1^2 + y_2^2)^2} \\
&+ i \Bigg\{ - \frac{2x_2}{x_2^2 + y_1^2} + \frac{2x_2}{x_2^2 + y_2^2} - \frac{2x_1}{x_1^2 + y_2^2} \\
&+ \frac{2x_1}{x_1^2 + y_1^2} + \frac{x_2}{x_2^2 + y_4^2} - \frac{x_2}{x_2^2 + y_3^2} \\
&+ \frac{x_1}{x_1^2 + y_3^2} - \frac{x_2\,(x_2^2 - y_2^2) + 2x_2\, y_2\, y_4}{(x_2^2 + y_2^2)^2} \\
&- \frac{x_1}{x_1^2 + y_4^2} + \frac{x_1\,(x_1^2 - y_2^2) + 2y_4\, x_1\, y_2}{(x_1^2 + y_2^2)^2} \\
&- \frac{x_1\,(x_1^2 - y_1^2) + 2y_3\, x_1\, y_1}{(x_1^2 + y_1^2)^2} \\
&+ \frac{x_2\,(x_2^2 - y_1^2) + 2y_3\, x_2\, y_1}{(x_2^2 + y_1^2)^2} \Bigg\} \Bigg].
\end{aligned}
\tag{57}
$$

Stresses in the matrix are determined directly from the functions $\phi'(z)$, $\phi''(z)$ and $\psi'(z)$ with the help of (1). At points of the inclusion these functions give only a part of the stresses. One has to add to them the stresses given in the equations (54) with $\delta_3 = 0$. The analysis can be given a check by verifying that the normal and shearing stresses are continuous across the boundary. Stresses in the matrix can be found from the following :

$$\frac{\sigma_x + \sigma_y}{2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)(1-\alpha)}{\pi(\alpha+1)} (\theta_1' - \theta_2' + \theta_3' - \theta_4')$$
$$+ \frac{\mu(\delta_1-\delta_2)}{\pi(\alpha-1)} \Big\{ \theta_1 - \theta_2 + \theta_3 - \theta_4 + \theta_1' - \theta_2'$$
$$+ \theta_3' - \theta_4' + \frac{4x_1(c+b)}{x_1^2+y_1^2} + \frac{4x_2(c-b)}{x_1^2+y_2^2}$$
$$- \frac{4x_1(c-b)}{x_1^2+y_2^2} - \frac{4x_2(c+b)}{x_2^2+y_1^2} \Big\},$$

$$\frac{\sigma_y - \sigma_x}{2} = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\pi(\alpha+1)} \Big[2(\alpha-1) \Big\{ \frac{yx_2}{x_2^2+y_1^2}$$
$$- \frac{yx_2}{x_2^2+y_2^2} + \frac{yx_1}{x_1^2+y_2^2} - \frac{x_1 y}{x_1^2+y_1^2} \Big\}$$
$$+ \frac{1-\alpha}{2} (\theta_1 - \theta_2 + \theta_3 - \theta_4 - \theta_1' + \theta_2' - \theta_3' + \theta_4') \Big]$$
$$+ \frac{\mu(\delta_1-\delta_2)}{\pi(\alpha+1)} \Big[\frac{2x_2(y+y_4)}{x_2^2+y_2^2}$$

$$- \frac{2x_2(y+y_3)}{x_2^2+y_1^2} + \frac{2x_1(y+y_3)}{x_1^2+y_1^2} - \frac{2x_1(y+y_4)}{x_1^2+y_2^2}$$

$$- \frac{2x_2 y(x_2^2-y_2^2)+4x_2y_2y_4y}{(x_2^2+y_2^2)^2}$$

$$+ \frac{2x_2 y(x_2^2-y_1^2)+4x_2yy_1y_3}{(x_2^2+y_1^2)^2}$$

$$- \frac{2x_1 y(x_1^2-y_1^2)+4x_1yy_1y_3}{(x_1^2+y_1^2)^2}$$

$$+ \frac{2yx_1(x_1^2-y_2^2)+4x_1yy_2y_4}{(x_1^2+y_2^2)^2}$$

$$+ \frac{2x_2y_4}{x_2^2+y_4^2} - \frac{2x_2y_3}{y_3^2+x_2^2} + \frac{2x_1y_3}{x_1^2+y_3^2}$$

$$\left. - \frac{2x_1y_4}{x_1^2+y_4^2} \right],$$

$$\tau_{xy} = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\pi(\alpha+1)} \left\{ \frac{-2\alpha yy_1-4yy_1-y_1y_3+x_2^2}{x_2^2+y_1^2} \right.$$

$$+ \frac{2\alpha yy_2+4yy_2+y_2y_4-x_2^2}{x_2^2+y_2^2}$$

$$+ \frac{-2\alpha yy_2-4yy_2-y_2y_4+x_1^2}{x_1^2+y_2^2}$$

$$+ \frac{2\alpha y y_1 + 4 y y_1 + y_1 y_3 - x_1^2}{x_1^2 + y_1^2}$$

$$+ \frac{\alpha - 1}{2} \log \frac{(x_2^2 + y_3^2)(x_1^2 + y_4^2)(x_2^2 + y_2^2)(x_1^2 + y_1^2)}{(x_1^2 + y_3^2)(x_2^2 + y_4^2)(x_2^2 + y_1^2)(x_1^2 + y_2^2)}$$

$$+ \frac{\mu(\delta_1 - \delta_2)}{\pi(\alpha + 1)} \left\{ \frac{x_2^2 + 4 y y_1 + y_1 y_3}{(x_2^2 + y_1^2)} + \frac{x_1^2 + 4 y y_2 + y_2 y_4}{x_1^2 + y_2^2} \right.$$

$$- \frac{x_1^2 + y_1 y_3 + 4 y y_1}{x_1^2 + y_1^2} - \frac{x_2^2 + y_2 y_4 + 4 y y_2}{x_2^2 + y_2^2}$$

$$- \frac{2 y y_4 (x_2^2 - y_2^2) - 4 x_2^2 y y_2}{(x_2^2 + y_2^2)^2} + \frac{x_2^2 - y_4^2}{x_2^2 + y_4^2}$$

$$+ \frac{2 y y_3 (x_2^2 - y_1^2) - 4 y y_1 x_2^2}{(x_2^2 + y_1^2)^2} - \frac{x_1^2 - y_4^2}{x_1^2 + y_4^2}$$

$$- \frac{2 y y_3 (x_1^2 - y_1^2) - 4 x_1^2 y y_1}{(x_1^2 + y_1^2)^2} + \frac{x_1^2 - y_3^2}{x_1^2 + y_3^2}$$

$$\left. + \frac{2 y y_4 (x_1^2 - y_2^2) - 4 x_1^2 y y_2}{(x_1^2 + y_2^2)^2} - \frac{x_2^2 - y_3^2}{x_2^2 + y_3^2} \right\}.$$

The case of shear when $\delta_1 = \delta_2 = 0$ and $\delta_3 \neq 0$ is dealt with in a similar fashion. The results are given below :

$$\phi'(z) = \frac{2\mu\delta_3}{\pi(\alpha+1)}\Big\{ \frac{1}{2}\log\frac{(x_2^2+y_2^2)(x_1^2+y_1^2)}{(x_1^2+y_2^2)(x_2^2+y_1^2)}$$

$$+\frac{1}{2}\log\frac{(x_2^2+y_3^2)(x_1^2+y_4^2)}{(x_1^2+y_3^2)(x_2^2+y_4^2)} - \frac{x_2^2+y_1y_3}{x_2^2+y_1^2}$$

$$+\frac{x_2^2+y_2y_4}{x_2^2+y_2^2} - \frac{x_1^2+y_2y_4}{x_1^2+y_2^2} + \frac{x_1^2+y_1y_3}{x_1^2+y_1^2}\Big\}$$

$$+\frac{2\mu\delta_3 i}{\pi(\alpha+1)}\Big\{ -\frac{1}{2}(\theta_1-\theta_2+\theta_3-\theta_4-\theta_1'$$

$$+\theta_2'-\theta_3'+\theta_4') + \frac{-x_2(y_1-y_3)}{x_2^2+y_1^2}$$

$$+\frac{x_2(y-y_4)}{x_2^2+y_4^2} - \frac{x_1(y_2-y_4)}{x_1^2+y_2^2} + \frac{x_1(y_1-y_3)}{x_1^2+y_1^2}\Big\},$$

$$\phi''(z) = \frac{2\mu\delta_3}{\pi(\alpha+1)}\Big[\frac{2x_2}{x_2^2+y_2^2} - \frac{2x_2}{x_2^2+y_1^2} + \frac{2x_1}{x_1^2+y_1^2}$$

$$-\frac{2x_1}{x_1^2+y_2^2} - \frac{x_2}{x_2^2+y_4^2} + \frac{x_2}{x_2^2+y_3^2}$$

$$-\frac{x_1}{x_1^2+y_3^2} + \frac{x_1}{x_1^2+y_4^2} - \frac{x_2(x_2^2-y_2^2)+2x_2y_2y_4}{(x_2^2+y_2^2)^2}$$

$$+\frac{x_2(x_2^2-y_1^2)+2y_3x_2y_1}{(y_1^2+x_2^2)^2}$$

$$- \frac{x_1(x_1^2 - y_1^2) + 2x_1 y_1 y_3}{(x_1^2 + y_1^2)^2} + \frac{x_1(x_1^2 - y_2^2) + 2 y_2 y_4 x_1}{(x_1^2 + y_2^2)^2}$$

$$+ i \Bigg\{ \frac{2y_1}{x_2^2 + y_1^2} - \frac{2y_2}{x_2^2 + y_2^2} + \frac{2y_2}{x_1^2 + y_2^2}$$

$$- \frac{2y_1}{x_1^2 + y_1^2} + \frac{y_4}{x_2^2 + y_4^2} - \frac{y_3}{x_2^2 + y_3^2} + \frac{y_3}{x_1^2 + y_3^2}$$

$$- \frac{y_4}{x_1^2 + y_4^2} - \frac{y_4(x_2^2 - y_2^2) - 2x_2^2 y_2}{(x_2^2 + y_2^2)^2}$$

$$+ \frac{y_3(x_2^2 - y_1^2) - 2x_2^2 y_1}{(x_2^2 + y_1^2)^2} - \frac{y_3(x_1^2 - y_1^2) - 2x_1^2 y_1}{(x_1^2 + y_1^2)^2}$$

$$+ \frac{y_4(x_1^2 - y_2^2) - 2x_1^2 y_2}{(x_1^2 + y_2^2)^2} \Bigg\} \Bigg],$$

$$\psi'(z) = \frac{2\mu\delta_3}{\pi(\alpha+1)} \Bigg[\frac{x_2(2x + x_2) + y_1(2y + y_3)}{(x_2^2 + y_1^2)}$$

$$- \frac{x_2(2x + x_2) + y_2(2y + y_4)}{(x_2^2 + y_2^2)}$$

$$+ \frac{x_1(2x + x_1) + y_2(2y + y_4)}{(x_1^2 + y_2^2)}$$

$$- \frac{x_1(2x+x_1)+y_1(2y+y_3)}{x_1^2+y_1^2}$$

$$- \frac{(x_2^2-y_1^2)(x_2x-yy_3)+2y_1x_2(x_2y+xy_3)}{(x_2^2+y_1^2)^2}$$

$$+ \frac{(x_2^2-y_2^2)(x_2x-yy_4)+2x_2y_2(x_2y+xy_4)}{(x_2^2+y_2^2)^2}$$

$$- \frac{(x_1^2-y_2^2)(x_1x-yy_4)+2x_1y_2(x_1y+xy_4)}{(x_1^2+y_2^2)^2}$$

$$+ \frac{(x_1^2-y_1^2)(x_1x-yy_3)+2y_1x_1(x_1y+xy_3)}{(x_1^2+y_1^2)^2}$$

$$- \frac{ax_2-(c+b)y_3}{x_2^2+y_3^2} + \frac{ax_2-(c-b)y_4}{x_2^2+y_4^2}$$

$$- \frac{ax_1+(c+b)y_3}{x_1^2+y_3^2} + \frac{ax_1+y_4(c-b)}{x_1^2+y_4^2}$$

$$+i\left\{ \frac{1+\alpha}{2}(\theta_1+\theta_2+\theta_3+\theta_4)\right.$$

$$- \frac{y_1(2x+x_2)-x_2(2y+y_3)}{x_2^2+y_1^2}$$

$$- \frac{y_2(2x+x_1)-x_1(2y+y_4)}{x_1^2+y_2^2}$$

$$+ \quad \frac{(2x + x_2)\,y_2 - (2y + y_4)\,x_2}{x_2^2 + y_2^2}$$

$$+ \quad \frac{(2x + x_1)\,y_1 - (2y + y_3)\,x_1}{x_1^2 + y_1^2}$$

$$- \quad \frac{2x_2 y_2 (x x_2 - y y_4) - (x_2^2 - y_2^2)(x_2 y + x y_4)}{(x_2^2 + y_2^2)^2}$$

$$- \quad \frac{2x_1 y_1 (x_1 x - y y_3) - (x_1^2 - y_1^2)(x_1 y + x y_3)}{(x_1^2 + y_1^2)^2}$$

$$+ \quad \frac{2x_2 y_1 (x x_2 - y y_3) - (x_2^2 - y_1^2)(x_2 y + x y_3)}{(x_2^2 + y_1^2)^2}$$

$$+ \quad \frac{2x_1 y_2 (x_1 x - y y_4) - (x_1^2 - y_2^2)(x_1 y + x y_4)}{(x_1^2 + y_2^2)^2}$$

$$+ \quad \frac{a y_3 + x_2 (c+b)}{x_2^2 + y_3^2} \quad - \quad \frac{a y_4 + x_2 (c+b)}{x_2^2 + y_4^2}$$

$$+ \quad \frac{a y_3 - x_1 (c+b)}{x_1^2 + y_3^2} \quad - \quad \left.\left. \frac{a y_4 - (c-b)\,x_1}{x_1^2 + y_4^2} \right\} \right] .$$

The stresses at points of the matrix are given by

$$\frac{\sigma_x + \sigma_y}{2} = \frac{2\mu\delta_3}{\pi(\alpha+1)}\Bigg[\log\frac{(x_1^2+y_1^2)(x_2^2+y_2^2)(x_2^2+y_3^2)(x_1^2+y_4^2)}{(x_2^2+y_1^2)(y_2^2+x_1^2)(x_2^2+y_4^2)(x_1^2+y_3^2)}$$

$$-\frac{2(x_2^2+yy_3)}{x_2^2+y_1^2} + \frac{2(x_2^2+y_2y_4)}{x_2^2+y_2^2}$$

$$-\frac{2(x_1^2+y_2y_4)}{x_1^2+y_2^2} + \frac{2(x_1^2+y_1y_3)}{x_1^2+y_1^2}\Bigg],$$

$$\frac{\sigma_y - \sigma_x}{2} = \frac{2\mu\delta_3}{\pi(\alpha+1)}\Bigg[\frac{x_2^2+4y_1y+yy_3}{x_2^2+y_1^2} - \frac{x_2^2+yy_4+4yy_2}{x_2^2+y_2^2}$$

$$+\frac{x_1^2+4yy_2+yy_4}{x_1^2+y_2^2} - \frac{x_1^2+4yy_1+yy_3}{x_1^2+y_1^2}$$

$$-\frac{(x_2^2-y_2^2)\,2yy_4 - 4x_2^2yy_2}{(x_2^2+y_2^2)^2}$$

$$+\frac{2yy_3(x_2^2-y_1^2) - 4yy_1x_2^2}{(x_2^2+y_1^2)^2}$$

$$-\frac{2yy_3(x_1^2-y_1^2) - 4yy_1x_1^2}{(x_1^2+y_1^2)^2}$$

$$+\frac{2yy_4(x_1^2-y_2^2) - 4x_1^2yy_2}{(x_1^2+y_2^2)^2} - \frac{x_2^2-y_4^2}{x_2^2+y_4^2}$$

$$+\frac{x_2^2-y_3^2}{x_2^2+y_3^2} - \frac{x_1^2-y_3^2}{x_1^2+y_3^2} + \frac{x_1^2-y_4^2}{x_1^2+y_4^2}\Bigg],$$

$$
\begin{aligned}
\tau_{xy} = \frac{2\mu\delta_3}{\pi(\alpha+1)} \Bigg[& \frac{2x_2(y+y_3)}{x_2^2+y_1^2} - \frac{2x_2(y+y_4)}{x_2^2+y_2^2} \\
& + \frac{2x_1(y+y_4)}{x_1^2+y_2^2} - \frac{2x_1(y+y_3)}{x_1^2+y_1^2} \\
& + \frac{2x_2y_4}{x_2^2+y_4^2} - \frac{2x_2y_3}{x_2^2+y_3^2} + \frac{2x_1y_3}{x_1^2+y_3^2} \\
& - \frac{2x_1y_4}{x_4^2+y_4^2} + \frac{2yx_2(x_2^2-y_2^2)+4x_2y_2yy_4}{(x_2^2+y_2^2)^2} \\
& - \frac{2yx_2(x_2^2-y_1^2)+4yy_1y_3x_2}{(x_2^2+y_1^2)^2} \\
& + \frac{2x_1y(x_1^2-y_1^2)+4x_1yy_1y_3}{(x_1^2+y_1^2)^2} \\
& - \frac{2yx_1(x_1^2-y_2^2)+4x_1yy_2y_4}{(x_1^2+y_2^2)^2} \Bigg].
\end{aligned}
$$

The stresses at points inside the inclusion could be found a similar way, keeping in mind the initial stresses present in it given by (54) with $\delta_1=\delta_2=0$.

Numerical work was done on the IBM 1620 computor. Lines of maximum shearing stress were drawn in the matrix for three sizes of the rectangle viz., 1 x 1, 2 x 1 and 10x 1 with distance c of the centre of the rectangle equal to 1, for the cases $\delta_1 = \delta_2 = \delta$ and $\delta_1 = -\delta_2 = \delta$. They are shown in Figs. 28-30 p. 147-149.

The non-vanishing component of stress σ_x along the leading edge y = 0 was also computed for these cases and also for pure shear case $\delta_1 = \delta_2 = 0$, $\delta_3 \neq 0$. They are shown in the Figs. 25-27 p. 145, 146.

CHAPTER VI

INHOMOGENEITY AND A POINT-FORCE ON THE BOUNDARY

Hitherto a point-force was acting in an otherwise homogeneous medium and we have used the complex potential functions associated with it. Expressions for these were available in the references ((2)) and ((3)). These expressions were useful for the solutions of the problems dealt with in earlier chapters. With a view to solve other problems. It is necessary to find the point-force effects for some more cases. This itself may henceforth be taken as an auxiliary, but an important problem in elasticity. We propose to find out the solution to the following problem :

A circular inhomogeneity of an elastic material present in an otherwise infinite, homogeneous, isotropic elastic medium called matrix. The elastic properties of the matrix are different from those of the inhomogeneity. A perfect bond is assumed to exist always between the inhomogeneity and the matrix. The centre of inhomogeneity may be taken to be the origin. A point-force P acts at a point ζ on their common boundary. The configuration is given in Fig. 14 p. 140 . It is required to find the state of stress and strain in the medium.

Let the radius of the inhomogeneity be a , so that the region $z\bar{z} \leq a^2$ of the complex plane represents it and the remaining portion of the plane represents the matrix. To distinguish between the points of the boundary and other points we take σ to be the boundary value of z . The equation of the boundary is therefore $\sigma\bar{\sigma} = a^2$. The following conditions must be satisfied by the final solution .

a) The normal and tangential stress should be continuous across the boundary.

b) The stresses should vanish at infinity.

c) The displacements at the boundary should be continuous and should be single-valued.

d) On physical grounds, at large distances the inhomogeneity should not affect the elastic fields of the matrix.

The construction of the potential functions is achieved in the following way. Assume the forms of the functions as under

$$\sigma_r - i\tau_{r\theta} = \phi'(z) + \overline{\phi'(z)} - e^{2i\theta}\left\{ \bar{z}\phi''(z) + \psi'(z) \right\}, \quad (63)$$

where θ is the vectorial angle. Using (58 - 62), (63) gives the following equation

$$\left[(\bar{\sigma}-\bar{\zeta})(2a^2 - \bar{\sigma}\zeta - \sigma\bar{\zeta}) + \sigma(\bar{\sigma}-\bar{\zeta})^2 \right](PE - PA - P)$$
$$+ \bar{P}(\bar{E} - 1 - \bar{A})(\sigma - \zeta)(2a^2 - \sigma\bar{\zeta} - \bar{\sigma}\zeta) + \left\{ \frac{1}{\sigma}(2PF\right.$$
$$\left. + 2\frac{K\bar{P}}{a^2}) + \frac{J\bar{P}}{a^2} - BP - \bar{B}\bar{P} + \frac{\sigma}{a^2}(\bar{P}F - \bar{P}I) \right\}\left\{ 6a^4\right.$$
$$\left. \frac{a^4\zeta^2}{\sigma^2} + \sigma^2\bar{\zeta}^2 - 4a^2\sigma\bar{\zeta} - \frac{4a^4\zeta}{\sigma} \right\} + (P\bar{\zeta} + \bar{P}H - \bar{P}D)$$
$$\times(\bar{\sigma}-\bar{\zeta})^2\frac{\sigma^2}{a^2} - \bar{P}\frac{\sigma^2}{a^2}(G + \alpha_m - C)(\bar{\sigma}-\bar{\zeta})(2a^2 - \bar{\sigma}\zeta - \sigma\bar{\zeta}) = 0.$$

Using $\sigma\bar{\sigma} = a^2$, the left hand side of the above equation can be put in the form of a polynomial in σ. Equating to zero the coefficients of various powers of σ, we get a set of seven linear equations in the various constants. It may be seen that only five are independent and are given below.

$$PK + \frac{\bar{P}K}{a^2} = 0,$$
$$-P(E - 1 - A) + \left(\frac{J\bar{P}}{a^2} - BP - \bar{B}\bar{P}\right)\zeta = 0,$$
$$\bar{E} - \bar{A} + \bar{F} - I - 1 = 0,$$
$$\bar{F} - I - G + C - \alpha_m = 0,$$
$$\left(\frac{J\bar{P}}{a^2} - BP - \bar{B}\bar{P}\right)a^2 + (P\bar{\zeta} + \bar{P}H - D\bar{P}) = 0.$$

Since the displacements are continuous across the boundary, therefore

$$(u+iv)_{ih}\Big|_{z=\sigma} = (u+iv)_m\Big|_{z=\sigma} \tag{64}$$

or

$$\frac{1}{\mu_{ih}}\left\{\alpha_{ih}\phi_{ih}(\sigma) - \sigma\,\overline{\phi'_{ih}(\sigma)} - \overline{\psi_{ih}(\sigma)}\right\} = \frac{1}{\mu_m}\left\{\alpha_m\phi_m(\sigma) - \sigma\overline{\phi'_m(\sigma)} - \overline{\psi_m(\sigma)}\right\}. \tag{65}$$

Using equation (58 - 61), equation (65) would yield the following equation

$$\begin{aligned}
&\beta\Big[\bar{P}\alpha_m\left\{-\log(\bar{\sigma}-\bar{\xi})(1-\bar{E}) + \bar{F}\log\bar{\sigma}\right\} \\
&\quad - \bar{\sigma}P\left\{(E-1)(\sigma-\xi)^{-1} + \frac{E}{\sigma}\right\} - \bar{P}\Big\{\alpha_m\log(\sigma-\xi) \\
&\quad + \bar{\xi}\frac{P}{\bar{P}}(\sigma-\xi)^{-1} + G\log(\sigma-\xi) + H(\sigma-\xi)^{-1} \\
&\quad + I\log\sigma + \frac{J}{\sigma} + \frac{K}{\sigma^2}\Big\}\Big] - \bar{P}\alpha_{ih}\left\{\bar{A}\log(\bar{\sigma}-\bar{\xi})\right. \\
&\quad \left. + \bar{\sigma}\bar{B}\right\} + \bar{\sigma}P\left\{A(\sigma-\xi)^{-1} + B\right\} + \bar{P}\left\{C\log(\sigma-\xi)\right. \\
&\quad \left. + D(\sigma-\xi)^{-1}\right\} = 0,
\end{aligned}$$

where $\beta = \mu_{ih}/\mu_m$. The symbol β becomes a non-dimensional parameter and gives the ratio of rigidity moduli of the inhomogeneity and the matrix. The terms containing logarithm and the remaining terms would vanish amongst themselves. The condition of vanishing of logarithmic terms yields the following two equations

The condition of vanishing of the remaining terms gives two more equations which are as follows :

$$a^2 P(\beta E - \beta - A) - (\beta J \bar{P} + \bar{P} \alpha_{ik} a^2 \bar{B} - a^2 PB)\zeta = 0,$$

$$(\beta - P\bar{\zeta} + PH) + (\beta J \bar{P} + \bar{P} \alpha_{ik} a^2 \bar{B} - a^2 PB - PD) = 0.$$

The restrictions that displacements be single valued in the matrix yields the equation

$$\alpha_m \bar{E} + \alpha_m \bar{F} + G + I = 0.$$

Finally from the last condition we get the following equations :

$$E + F = 0,$$

and

$$G + I = 0.$$

The set twelve equations listed above for eleven constants A, B, ..., K consistant. For case, we write $\nu_1 = (1-\beta)/(\beta\alpha_m + 1)$, $\nu_2 = (\alpha_{ik} - \beta\alpha_m)/(\beta + \alpha_{ik})$. Thus

$$A = \nu_2 - 1,$$

$$B = \frac{(\nu_2-1)(\beta-1)}{2a^2 P}\left\{\frac{\bar{P}\zeta + P\bar{\zeta}}{2\beta + \alpha_{ik} - 1} + \frac{\bar{P}\zeta - P\bar{\zeta}}{\alpha_{ik}+1}\right\},$$

$$C = (1-\nu_1)\alpha_m, \qquad D = \frac{P}{\bar{P}}\bar{\zeta}(1-\nu_2),$$

$$E = -\nu_1\alpha_m, \qquad F = \nu_1\alpha_m, \qquad G = \nu_2 = -I$$

$$H = \frac{P}{\bar{P}}\zeta\,\nu_1\alpha_m, \qquad K = -\nu_1\alpha_m a^2\frac{P}{\bar{P}}$$

$$J = -\frac{1}{\bar{P}}\left\{P\bar{\zeta}(\nu_1\alpha_m + \nu_2) + \frac{(\bar{P}\zeta + P\bar{\zeta})(\nu_2-1)(1-\beta)}{2\beta + \alpha_{ik} - 1}\right\}.$$

Thus the effect of an isolated force P acting on a fixed point ζ of the boundary $\sigma\bar{\sigma} = a^2$ of the inhomogeneity is given by the following functions:

$$
\begin{aligned}
\phi_{in}(z) = {} & \frac{P}{2\pi(\alpha_m+1)} \Big\{ (\nu_2 - 1) \log(z-\zeta) \\
& - \frac{(\nu_2-1)(\beta-1)^2 z \bar{\zeta}}{a^2(2\beta+\alpha_{in}-1)(\alpha_{in}+1)} \Big\} \\
& + \frac{\bar{P}}{2\pi(\alpha_m+1)} \frac{\zeta z}{a^2} \Big\{ \frac{(\nu_2-1)(\beta-1)(\beta+\alpha_{in})}{(2\beta+\alpha_{in}-1)(\alpha_{in}+1)} \Big\},
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\psi_{in}(z) = {} & \frac{\bar{P}}{2\pi(\alpha_m+1)} \Big\{ (1-\nu_1)\alpha_m \log(z-\zeta) \Big\} \\
& + \frac{P}{2\pi(\alpha_m+1)} \Big\{ (1-\nu_2)\bar{\zeta}(z-\zeta)^{-1} \Big\},
\end{aligned}
\tag{67}
$$

$$
\phi_m(z) = \frac{P}{2\pi(\alpha_m+1)} \Big\{ -(1+\nu_1\alpha_m)\log(z-\zeta) + \nu_1\alpha_m \log z \Big\},
\tag{68}
$$

$$
\begin{aligned}
\psi_m(z) = {} & \frac{\bar{P}}{2\pi(\alpha_m+1)} \Big\{ (\nu_2+\alpha_m)\log(z-\zeta) - \nu_2 \log z \\
& + \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{in}-1} \zeta z^{-1} \Big\} + \frac{P}{2\pi(\alpha_m+1)} \Big\{ (1+\nu_1\alpha_m) \\
& \times \bar{\zeta}(z-\zeta)^{-1} - \Big(\nu_1\alpha_m + \nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{in}-1} \Big) \\
& \times \bar{\zeta} z^{-1} - \nu_1 \alpha_m a^2 z^{-2} \Big\}.
\end{aligned}
\tag{69}
$$

The effect of a concentrated force acting on the boundary when there is a cavity may easily be obtained by simply setting $\beta = 0,\ \nu_1 = \nu_2 = 1$. Various other results may be obtained, e.g. if the inhomogeneity and the matrix are of the same materials, one obtains the results for the case of a point-force acting in an infinite two dimensional medium by

putting $\beta = 1$, $\nu_1 = \nu_2 = 0$. If the inhomogeneity is rigid, one sets $\beta \to \infty$, $\nu_1 = -\frac{1}{\alpha_m}$ and $\nu_2 = -\alpha_m$. Some of these particular cases have been dealt with in the books on elasticity theory ((2)) and ((3)). In the next chapter the results of this chapter are used for solving the problem of circular inclusion whose elastic constants differ from those of the matrix. This problem has been done by many authors ((16)). Eshelby also indicated a method of solving the inhomogeneity problems as particular cases of inclusion problem , but the argument is very complicated. The purpose of solving this known problem is to indicate a direct method of solving such problems by using the results of this chapter.

CHAPTER VII

CIRCULAR INCLUSION WITH DIFFERENT ELASTIC MODULI

The mathematical model of the problem considered in this chapter is as follows. The circular inhomogeneity of chapter VI tends to undergo a deformation which would be uniform in the absence of the matrix. If this tendency is opposed the following system of stresses would develop in the inclusion. It may be noted that we are using the subscripts ih and m to distinguish quantities pertaining to the inhomogeneity and the matrix respectively.

$$\left.\begin{aligned} \sigma_x^0 &= -\{\lambda_{ih}(\delta_1+\delta_2)+2\mu_{ih}\delta_1\}, \\ \sigma_y^0 &= -\{\lambda_{ih}(\delta_1+\delta_2)+2\mu_{ih}\delta_2\}, \\ \tau_{xy}^0 &= -\{2\mu_{ih}\delta_3\}. \end{aligned}\right\} \quad (70)$$

The point-force layer generated by the deforming inclusion is given by

$$P\,ds = -i\,(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\,d\zeta + i\,\mu_{ih}(\delta_1-\delta_2-2i\delta_3)\,d\bar{\zeta},$$

$$\bar{P}\,ds = i\,(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\,d\bar{\zeta} - i\,\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\,d\zeta. \qquad (71)$$

The effect of single point force P acting at a point ζ of the boundary $\tau\bar{\tau}=a^2$ was found in chapter VI , equations (66 - 69) . Thus for a continuous distribution of point-forces given by equation (71), it is easily seen that the following relations giving the cumulative effect of the distribution are true.

$$\begin{aligned}
\phi'_{ih}(z) = & \frac{-i\,(\nu_2-1)(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1}\,d\zeta \\
& + \frac{i\,\mu_{ih}(\nu_2-1)(\delta_1-\delta_2+2i\delta_3)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1}\,d\bar{\zeta} \\
& + \frac{i(\nu_2-1)(\beta-1)^2(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)}{2\pi a^2(\alpha_{ih}+1)(\alpha_m+1)(2\beta+\alpha_{ih}-1)} \int_\gamma \bar{\zeta}\,d\zeta \\
& - \frac{i(\nu_2-1)(\beta-1)^2\mu_{ih}(\delta_1-\delta_2-2i\delta_3)}{(\alpha_{ih}+1)\,2\pi a^2(\alpha_m+1)(2\beta+\alpha_{ih}-1)} \int_\gamma \bar{\zeta}\,d\bar{\zeta} \\
& + \frac{i(\nu_2-1)(\beta-1)(\beta+\alpha_{ih})(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)}{2\pi a^2(\alpha_m+1)(2\beta+\alpha_m-1)(\alpha_{ih}+1)} \int_\gamma \zeta\,d\bar{\zeta} \\
& - \frac{i\,\mu_{ih}(\nu_2-1)(\beta-1)(\beta+\alpha_{ih})(\delta_1-\delta_2-2i\delta_3)}{2\pi a^2(\alpha_m+1)(2\beta+\alpha_m-1)(\alpha_{ih}+1)} \int_\gamma \zeta\,d\zeta ,
\end{aligned}$$

$$\Psi'_{ih}(z) = \frac{i(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)(1-\nu_1)\alpha_m}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\bar{\zeta}$$

$$- \frac{i\mu_{ih}(\delta_1-\delta_2-2i\delta_3)(1-\nu_1)\alpha_m}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\zeta$$

$$+ \frac{i(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)(1-\nu_2)}{2\pi(\alpha_m+1)} \int_\gamma \bar{\zeta}(z-\zeta)^{-2} d\zeta$$

$$- \frac{i\mu_{ih}(1-\nu_2)(\delta_1-\delta_2+2i\delta_3)}{2\pi(\alpha_m+1)} \int_\gamma \bar{\zeta}(z-\zeta)^{-2} d\bar{\zeta} ,$$

$$\Phi'_m(z) = \frac{i(1+\nu_1\alpha_m)(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\zeta$$

$$- \frac{i(1+\nu_1\alpha_m)\mu_{ih}(\delta_1-\delta_2+2i\delta_3)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\bar{\zeta}$$

$$- \frac{i(\delta_1+\delta_2)\nu_1\alpha_m(\lambda_{ih}+\mu_{ih})}{2\pi(\alpha_m+1)} \int_\gamma z^{-1} d\zeta$$

$$+ \frac{i\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\nu_1\alpha_m}{2\pi(\alpha_m+1)} \int_\gamma z^{-1} d\bar{\zeta}$$

$$\Psi'_m(z) = \frac{i(\alpha_m+\nu_2)(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\bar{\zeta}$$

$$- \frac{i\mu_{ih}(\nu_2+\alpha_m)(\delta_1-\delta_2-2i\delta_3)}{2\pi(\alpha_m+1)} \int_\gamma (z-\zeta)^{-1} d\zeta$$

$$- \frac{i(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\nu_2}{2\pi(\alpha_m+1)} \int_\gamma z^{-1} d\bar{\zeta}$$

$$+ \frac{i\mu_{ih}(\delta_1-\delta_2-2i\delta_3)\nu_2}{2\pi(\alpha_m+1)} \int_\gamma z^{-1} d\zeta$$

$$+ \frac{i(\nu_2-1)(1-\beta)(\delta_1+\delta_2)(\lambda_{ih}+\mu_{ih})}{2\pi(\alpha_m+1)(\alpha_{ih}+2\beta-1)} \int_\gamma \zeta z^{-2} d\bar{\zeta}$$

$$- \frac{i\mu_{ih}(\nu_2-1)(\beta-1)(\delta_1-\delta_2-2i\delta_3)}{2\pi(\alpha_m+1)(2\beta+\alpha_{ih}-1)} \int_\gamma \zeta z^{-2} d\zeta \; +$$

$$+ \frac{i(\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2)(1 + \nu_1 \alpha_m)}{2\pi(\alpha_m + 1)} \int_\gamma \bar{\zeta}\,(z - \zeta)^{-2}\, d\zeta$$

$$- \frac{i \mu_{ih}(\delta_1 - \delta_2 + 2i\delta_3)(1 + \nu_1 \alpha_m)}{2\pi(\alpha_m + 1)} \int_\gamma \bar{\zeta}\,(z - \zeta)^{-2}\, d\bar{\zeta}$$

$$- \frac{i(\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2)}{2\pi(\alpha_m + 1} \left(\nu_1 \alpha_m + \nu_2 + \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right) \int_\gamma \bar{\zeta}\, z^{-2}\, d\zeta$$

$$+ \frac{i \mu_{ih}(\delta_1 - \delta_2 + 2i\delta_3)}{2\pi(\alpha_m + 1)} \left(\nu_1 \alpha_m + \nu_2 + \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right) \int_\gamma \bar{\zeta}\, z^{-2}\, d\bar{\zeta}$$

$$- \frac{i \nu_1 \alpha_m (\delta_1 + \delta_2)(\lambda_{ih} + \mu_{ih})}{2\pi(\alpha_m + 1)} \int_\gamma z^{-3}\, d\zeta$$

$$+ \frac{i \nu_1 \alpha_m \mu_{ih}(\delta_1 - \delta_2 + 2i\delta_3)}{2\pi(\alpha_m + 1)} \int_\gamma z^{-3}\, d\bar{\zeta} ,$$

where γ is the contour of the inclusion. The integrals can be evaluated by making use of the relations $\bar{\zeta} = a^2/\zeta$ and $d\bar{\zeta} = -\frac{a^2}{\zeta^2} d\zeta$ on γ , wherever necessary. The resulting potentials are given below .

$$\phi'_{ih}(z) = \frac{\beta}{2\beta + \alpha_{ih} - 1} (\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2), \tag{72}$$

$$\psi'_{ih}(z) = \frac{(\nu_1 - 1)\alpha_m}{\alpha_m + 1} \mu_{ih} (\delta_1 - \delta_2 - 2i\delta_3), \tag{73}$$

$$\phi'_m(z) = -\frac{1 + \nu_1 \alpha_m}{\alpha_m + 1} \mu_{ih} \frac{a^2}{z^2} (\delta_1 - \delta_2 + 2i\delta_3), \tag{74}$$

$$\psi'_m(z) = \frac{\alpha_{ih} - 1}{2\beta + \alpha_{ih} - 1} (\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2) \frac{a^2}{z^2} - \frac{\mu_{ih}(1 + \nu_1 \alpha_m)}{\alpha_m + 1} (\delta_1 - \delta_2 + 2i\delta_3) \frac{3a^4}{z^4} . \tag{75}$$

The stress field inside and outside the inclusion can be evaluated from the appropriate functions by means of relations (1). It must however be remembered that the inhomogeneity had a constrained-stress field given by (70) and it must be added to the stress field obtained from these functions. Continuity of normal and shearing stresses can be verified at this stage. The stress components in polar form are given below.

$$(\sigma_r)_{ih} = \frac{1-\alpha_{ih}}{2\beta+\alpha_{ih}-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}$$
$$\times(\delta_1-\delta_2)\cos 2\theta - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}\, 2\delta_3 \sin 2\theta,$$

$$(\sigma_\theta)_{ih} = \frac{1+\alpha_{ih}}{2\beta+\alpha_i-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) + \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2)\cos 2\theta$$
$$+\ 2\mu_{ih}\,\delta_3 \sin 2\theta\left(\frac{1+\nu_1\alpha_m}{\alpha_m+1}\right),$$

$$(\tau_{r\theta})_{ih} = \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2)\sin 2\theta - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\, 2\mu_{ih}\cos 2\theta,$$

$$(\sigma_r)_m = \frac{1-\alpha_{ih}}{2\beta+\alpha_{ih}-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\frac{a^2}{r^2} + \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2)$$
$$\times\left(-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta + 2\mu_{ih}\left(\frac{1+\nu_1\alpha_m}{\alpha_m+1}\right)$$
$$\times\,\delta_3\frac{a^2}{r^2}\left(-4+\frac{3a^2}{r^2}\right)\sin 2\theta,$$

$$(\sigma_\theta)_m = -\frac{1-\alpha_{ih}}{2\beta+\alpha_i-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\frac{a^2}{r^2} - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\frac{3a^4}{r^4}$$
$$\times\,\mu_{ih}(\delta_1-\delta_2)\cos 2\theta - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}\frac{6a^4}{r^4}\sin 2\theta,$$

$$(\tau_{r\theta})_m = -\frac{1+\nu_1\alpha_m}{1+\alpha_m}\mu_{ih}(\delta_1-\delta_2)\left(2-\frac{3a^2}{r^2}\right)\frac{a^2}{r^2}\sin 2\theta$$
$$+\ \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}\,\delta_3\left(2-\frac{3a^2}{r^2}\right)\frac{2a^2}{r^2}\cos 2\theta.$$

The results obtained in this chapter agree with those obtained by other writers ((16)).

Integration of the functions given in (72 - 75) with respect to z yields

$$\phi_{ih}(z) = \frac{\beta}{2\beta + \alpha_{ih} - 1}(\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2)z, \tag{76}$$

$$\psi_{ih}(z) = \frac{(\nu_1 - 1)\alpha_m}{\alpha_m + 1}\mu_{ih}(\delta_1 - \delta_2 - 2i\delta_3)z, \tag{77}$$

$$\phi_m(z) = \frac{1 + \nu_1\alpha_m}{\alpha_m + 1}\mu_{ih}\frac{a^2}{z}(\delta_1 - \delta_2 + 2i\delta_3), \tag{78}$$

$$\psi_m(z) = \frac{1 - \alpha_{ih}}{2\beta + \alpha_{ih} - 1}(\lambda_{ih} + \mu_{ih})(\delta_1 + \delta_2)\frac{a^2}{z} + \mu_{ih}\left(\frac{1 + \nu_1\alpha_m}{\alpha_m + 1}\right)(\delta_1 - \delta_2 + 2i\delta_3)\frac{a^4}{z^3}. \tag{79}$$

The displacement field inside and outside can be evaluated from appropriate potentials with the help of (2). The displacements on the boundary can be seen to be

$$u_r\Big|_{z=\sigma} = \frac{a\beta(\alpha_{ih} - 1)}{2(2\beta + \alpha_{ih} - 1)}\left(\frac{\lambda_{ih}}{\mu_{ih}} + 1\right)(\delta_1 + \delta_2) + \frac{(1 - \nu_1)\alpha_m}{\alpha_m + 1}(\delta_1 - \delta_2)\,a\cos 2\theta + \frac{(1 - \nu_1)\alpha_m}{\alpha_m + 1}2\delta_3\,a\sin 2\theta,$$

$$(u_\theta)\Big|_{z=\sigma} = \frac{(1 - \nu_1)\alpha_m}{\alpha_m + 1}(\delta_1 - \delta_2)\,a\sin 2\theta - \frac{(1 - \nu_1)\alpha_m}{\alpha_m + 1}2\delta_3\,a\cos 2\theta.$$

C H A P T E R VIII

A CIRCULAR HOLE AND A POINT-FORCE

In this Chapter another important problem is considered. This relates to the case, when there is a circular hole in the body. A concentrated force is applied at a point of the body. The elastic field generated by its action is to be found out. Choosing the coordinate system with origin at the centre of the hole, (Fig. 15 p. 140) we have $z\bar{z} = a^2$ as the equation of the boundary of the hole. Assume the forms of the complex functions which describe the effect of the concentrated force P acting at the point ζ as under :

$$\phi(z) = \frac{P}{2\pi(\alpha+1)} \Big\{ -\log(z-\zeta) - \alpha \log\left(z-\frac{a^2}{\bar{\zeta}}\right)$$

$$+ \quad A\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1} + \alpha \log z \Big\}, \qquad (80)$$

$$\Psi(z) = \frac{\bar{P}}{2\pi(\alpha+1)} \Big\{ \alpha \log(z-\zeta) + \bar{\zeta}\frac{P}{\bar{P}}(z-\zeta)^{-1}$$
$$+ \log\left(z - \frac{a^2}{\bar{\zeta}}\right) + B\left(z - \frac{a^2}{\bar{\zeta}}\right)^{-1} + C\left(z - \frac{a^2}{\bar{\zeta}}\right)^{-2}$$
$$- \log z + D z^{-1} + E z^{-2} \Big\}, \tag{81}$$

where $\bar{P}$ is the complex conjugate of P. Here we have introduced the unknown constants A,B,C,D and E. This choice gives a proper singularity at the point ζ of action of the concentrated force. The unknown constants have to be determined using appropriate boundary conditions. Thus vanishing of normal and shearing stresses on the free edge of the hole implies that

$$(\sigma_r - i\tau_{r\theta})_{z=\sigma} = 0$$

We use the relation (63) viz.,

$$(\sigma_r - i\tau_{r\theta}) = \phi'(z) + \overline{\phi'(z)} - e^{2i\theta}\left[\bar{z}\phi''(z) + \psi'(z)\right].$$

Substituting the values of $\phi'(z)$ and $\psi'(z)$ from (80) and (81), the above equation leads to

$$- P(\sigma-\zeta)^{-1} + \bar{\zeta}\frac{P\alpha}{\sigma}(\bar{\sigma}-\bar{\zeta})^{-1} - PA\frac{\bar{\zeta}^2}{\sigma^2} + \frac{2P\alpha}{\sigma} - \bar{P}(\bar{\sigma}-\bar{\zeta})^{-1}$$
$$+ \bar{P}\frac{\alpha\zeta}{\bar{\sigma}}(\sigma-\zeta)^{-1} - \bar{P}\bar{A}\frac{\zeta^2}{\bar{\sigma}^2}(\sigma-\zeta)^{-2} + \frac{\bar{P}\alpha}{\bar{\sigma}} - P\bar{\sigma}(\sigma-\zeta)^{-2}$$
$$- P\frac{\alpha\bar{\zeta}^2}{\sigma}(\bar{\sigma}-\bar{\zeta})^{-2} + 2\frac{PA}{\sigma^2}\bar{\zeta}^3(\bar{\sigma}-\bar{\zeta})^{-3} - \frac{\bar{P}\alpha}{a^2}\sigma^2(\sigma-\zeta)^{-1}$$
$$+ P\bar{\zeta}\frac{\sigma^2}{a^2}(\sigma-\zeta)^{-2} + B\bar{P}\frac{\bar{\zeta}^2}{a^2}(\bar{\sigma}-\bar{\zeta})^{-2} + \sigma\bar{\zeta}\frac{\bar{P}}{a^2}(\bar{\sigma}-\bar{\zeta})^{-1}$$
$$- 2C\frac{\bar{P}}{\sigma}\frac{\bar{\zeta}^3}{a^2}(\bar{\sigma}-\bar{\zeta})^{-3} + \frac{\sigma\bar{P}}{a^2} + 2\frac{\bar{P}E}{a^2\sigma} + \frac{\bar{P}D}{a^2} = 0.$$

The left side of this equation may be put in the form of a polynomial in σ. On equating the coefficients of various powers of σ to zero, we obtain the following equations

$$\bar{P} C + \bar{P} \zeta B = 0,$$

$$\bar{P} E + P \alpha a^2 = 0,$$

$$P a^4 \zeta + \alpha P b^2 a^2 \zeta - P A b^4 + D \bar{P} a^2 \zeta^2 = 0,$$

$$P \bar{\zeta}^3 - \bar{A} \bar{P} \frac{b^4}{a^2} + \bar{P} B \bar{\zeta}^2 + D \bar{P} \bar{\zeta}^2 = 0,$$

$$P \frac{A}{a^2} \bar{\zeta}^3 + \bar{P} \bar{A} \zeta \gamma^2 - \bar{P} B \bar{\zeta} \gamma^2 - D \bar{P} \bar{\zeta} (1-\gamma^2) - 2 P \bar{\zeta}^2 - \alpha P \bar{\zeta}^2 = 0,$$

$$-P (a^2+b^2)(a^2+b^2\alpha) + P A b^2 \bar{\zeta} + P A \frac{b^4}{a^2} \bar{\zeta} - D \bar{P} \zeta (a^2+b^2) = 0,$$

$$-(\bar{\zeta} + 4\gamma^2 \bar{\zeta}^2) P A + \bar{\zeta} P \alpha a^2 - \zeta^2 \bar{P} \bar{A} + D \bar{P} \left(a^2 + \frac{b^4}{a^2} + 4b^2\right) + \bar{P} J \frac{b^4}{a^2} + 4 b^2 \bar{\zeta} P \alpha + 5 a^2 \bar{\zeta} P + b^2 \bar{\zeta} P = 0,$$

where $b = |\zeta|$, $\gamma = b/a$.

Solving this set of equations we obtain

$$A = \frac{\bar{P}}{P} \frac{\gamma^2 - 1}{a^2 \gamma^6} \zeta^3, \qquad B = \frac{P}{\bar{P}} \alpha \bar{\zeta} - \frac{\gamma^2 - 1}{\gamma^2} \zeta,$$

$$C = \frac{\gamma^2 - 1}{\gamma^4} \zeta^2 \qquad D = \frac{\gamma^2 - 1}{\gamma^2} \zeta - \frac{P}{\bar{P}} \alpha \bar{\zeta} - \frac{P}{\bar{P}} \frac{\bar{\zeta}}{\gamma^2},$$

$$E = -\frac{P}{\bar{P}} \alpha a^2.$$

Thus, for an isolated point-force P acting at the point ζ, the complex potentials are

$$\Phi(z) = \frac{P}{2\pi(\alpha+1)}\left\{-\log(z-\zeta) - \alpha\log\left(z-\frac{a^2}{\bar{\zeta}}\right) + \alpha\log z\right\}$$
$$+\frac{\bar{P}}{2\pi(\alpha+1)}\,\frac{\gamma^2-1}{a^2\gamma^6}\,\zeta^3\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1}, \qquad (82)$$

$$\Psi(z) = \frac{\bar{P}}{2\pi(\alpha+1)}\left\{\alpha\log(z-\zeta) + \log\left(z-\frac{a^2}{\bar{\zeta}}\right) - \log z\right.$$
$$-\frac{\gamma^2-1}{\gamma^2}\,\zeta\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1} + \frac{\gamma^2-1}{\gamma^4}\,\zeta^2\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-2}$$
$$\left.+\frac{\gamma^2-1}{\gamma^2}\,\frac{\zeta}{z}\right\} + \frac{P}{2\pi(\alpha+1)}\left\{\alpha\bar{\zeta}\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1}\right.$$
$$\left.+\bar{\zeta}(z-\zeta)^{-1} - \alpha\bar{\zeta}z^{-1} - \alpha a^2 z^{-2} - \bar{\zeta}\frac{z^{-1}}{\gamma^2}\right\}. \qquad (83)$$

From (82) and (83) we get by simple differentiation

$$\Phi'(z) = \frac{P}{2\pi(\alpha+1)}\left\{(\zeta-z)^{-1} - \alpha\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1} + \alpha z^{-1}\right\}$$
$$-\frac{\bar{P}}{2\pi(\alpha+1)}\,\zeta^3\,\frac{\gamma^2-1}{a^2\gamma^6}\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-2}, \qquad (84)$$

$$\Psi'(z) = \frac{\bar{P}}{2\pi(\alpha+1)}\left\{\alpha(z-\zeta)^{-1} + \left(z-\frac{a^2}{\bar{\zeta}}\right)^{-1} - z^{-1} + \frac{\gamma^2-1}{\gamma^2}\zeta\right.$$
$$\left.\times\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-2} - 2\zeta^2\,\frac{\gamma^2-1}{\gamma^4}\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-3} - \frac{\gamma^2-1}{\gamma^2}\,\zeta z^{-2}\right\}$$
$$+\frac{P}{2\pi(\alpha+1)}\left\{-\bar{\zeta}(z-\zeta)^{-2} - \alpha\bar{\zeta}\left(z-\frac{a^2}{\bar{\zeta}}\right)^{-2}\right.$$
$$\left.+\alpha\bar{\zeta}z^{-2} + \frac{\bar{\zeta}}{\gamma^2}z^{-2} + 2\alpha a^2 z^{-3}\right\}. \qquad (85)$$

The associated elastic fields can be readily obtained with the help of (1). It shall not be done here, because that will be going beyond the main stream of this thesis, viz. inclusion problems. The results obtained above will be used in the next chapter, where we deal with an inclusion in the presence of a hole.

It can be readily seen that the cumulative effect of a continuous distribution of point-forces acting along any finite curve in elastic plane, in place of a single isolated force at ζ, could be described by the functions

$$\phi'(z) = \frac{1}{2\pi(\alpha+1)}\Big\{\int_\gamma P(\zeta - z)^{-1} ds + \int_\gamma \alpha P z^{-1} ds - \int_\gamma \alpha P \left(z - \frac{a^2}{\bar{\zeta}}\right)^{-1} ds - \int_\gamma \bar{P}\, \frac{\gamma^2 - 1}{a^2 \gamma^6}\, \zeta^3 \left(z - \frac{a^2}{\bar{\zeta}}\right)^{-2} ds\Big\}, \tag{86}$$

$$\begin{aligned}\psi'(z) = \frac{1}{2\pi(\alpha+1)}\Big\{ & \int_\gamma \alpha \bar{P}(z-\zeta)^{-1} ds + \int_\gamma \bar{P}\left(z - \frac{a^2}{\bar{\zeta}}\right)^{-1} ds \\ & - \int_\gamma z^{-1} \bar{P}\, ds + \int_\gamma \bar{P}\, \frac{\gamma^2 - 1}{\gamma^2}\, \zeta \left(z - \frac{a^2}{\bar{\zeta}}\right)^{-2} ds \\ & - \int_\gamma 2\bar{P}\, \frac{\gamma^2 - 1}{\gamma^4}\, \zeta^2 \left(z - \frac{a^2}{\bar{\zeta}}\right)^{-3} ds - \int_\gamma \bar{P}\, \frac{\gamma^2 - 1}{\gamma^2}\, \zeta z^{-2} ds \\ & - \int_\gamma P \bar{\zeta} (z-\zeta)^{-2} ds - \int_\gamma \alpha P \left(z - \frac{a^2}{\bar{\zeta}}\right)^{-2} ds \\ & + \int_\gamma \alpha P \bar{\zeta} z^{-2} ds + \int_\gamma \frac{P \bar{\zeta} z^{-2}}{\gamma^2} ds + \int_\gamma 2 P \alpha a^2 z^{-3} ds \Big\},\end{aligned} \tag{87}$$

where γ is the curve, ds is its differential arc length and ζ any point on it.

CHAPTER IX

CIRCULAR INCLUSION IN THE PRESENCE OF A CIRCULAR HOLE

A circular region in an infinite medium containing a circular cavity tends to undergo dimensional changes. The cavity and the inclusion do not over lap. The presence of the hole is expected to greatly perturb the elastic fields in the region of physical interest, especially when it is comparitively close to the inclusion.

Choosing the reference system in accordance with Fig. 10 p. 138 it is easy to keep in mind that $z\bar{z} \leq a^2$ and $(z-\ell)(\bar{z}-\ell) \leq 1$ represent the hole with centre at the origin and radius 'a' and the inclusion with centre at ℓ (real) and radius unity. The remaining portion of the complex plane represents the matrix.

The inclusion in the absence of the matrix, would undergo a displacement characterised by the components

$$u = \delta_1(x-l) + \delta_3 y,$$

$$v = \delta_2 y + \delta_3 (x-l).$$

As before if the above displacement is opposed, then the stress system developed in the inclusion region is

$$\left.\begin{aligned} \sigma_x^{\circ} &= -\{\lambda(\delta_1+\delta_2) + 2\mu\delta_1\}, \\ \sigma_y^{\circ} &= -\{\lambda(\delta_1+\delta_2) + 2\mu\delta_2\}, \\ \tau_{xy}^{\circ} &= -2\mu\delta_3, \end{aligned}\right. \tag{88}$$

and the point-force distribution generated along the contour calculated from (88) and (22) is given by

$$\left.\begin{aligned} P\,ds &= -i(\lambda+\mu)(\delta_1+\delta_2)\,d\zeta + i\mu(\delta_1-\delta_2+2i\delta_3)\,d\bar{\zeta} \\ &\text{and} \\ \bar{P}\,ds &= i(\lambda+\mu)(\delta_1+\delta_2)\,d\bar{\zeta} - i\mu(\delta_1-\delta_2-2i\delta_3)\,d\zeta \end{aligned}\right\} \tag{89}$$

Substituting these in the integrals (86), (87) and taking γ to be the boundary $(\zeta-l)(\bar{\zeta}-l) = 1$, we evaluate the integrals. Also there are the relations $\bar{\zeta} = l + (\zeta-l)^{-1}$ and $d\bar{\zeta} = -(\zeta-l)^{-2}\,d\zeta$ on γ. The expressions become particularly simple after the substitutions

$$z_1 = z - a^2/l,$$

and

$$z_2 = z - l.$$

Suffixes i and m will be used to distinguish quantities pertaining to the inclusion and the matrix respectively. The result

of integration is that

$$\Phi_i'(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} + \frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_1^2}$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\Big\{(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2}$$
$$+ (a^2-\ell^2+3)\frac{2a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4}\Big\}, \quad (90)$$

$$\Psi_i'(z) = \frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\left(\frac{2a^2}{\ell z_1^3} - \frac{1}{z_1^2} + \frac{1}{z^2}\right)$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\Big\{\alpha + \frac{a^2}{\ell^2 z^2} + \frac{2a^4}{\ell^3 z_1^3}$$
$$+ (2a^2-2\ell^2+3)\frac{3a^4}{\ell^4 z_1^4} + \frac{12a^6}{\ell^5 z_1^5}\Big\}$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{\ell^2 z^2}, \quad (91)$$

$$\Phi_m'(z) = \frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_1^2} - \frac{\mu}{\alpha+1}(\delta_1-\delta_2+2i\delta_3)\frac{1}{z_2^2}$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\Big[(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2}$$
$$+ (a^2-\ell^2+3)\frac{2a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4}\Big], \quad (92)$$

$$\Psi_m'(z) = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{\frac{1}{z_2^2} + \frac{2a^2}{\ell z_1^3} + \frac{1}{z^2} - \frac{1}{z_1^2}\Big\}$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\Big\{\frac{a^2}{\ell^2 z^2} + \frac{2a^4}{\ell^3 z_1^3}$$
$$+ (2a^2-2\ell^2+3)\frac{3a^4}{\ell^4 z_1^4} + \frac{12a^6}{\ell^5 z_1^5}\Big\}$$
$$- \frac{\mu}{\alpha+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{a^2}{\ell^2 z^2} + \frac{2\ell}{z_2^3} + \frac{3}{z_2^4}\right). \quad (93)$$

The stresses in the matrix are obtained directly from $\phi_m'(z)$ and $\psi_m'(z)$ by using the equation (1), but the inclusion had an initial stress field given by (88), therefore it has to be added to the one obtained from $\phi_i'(z)$ and $\psi_i'(z)$ given above. Moreover at this stage it can be verified that the boundary conditions on the stresses both at the hole $\sigma\bar{\sigma}=a^2$ and at the inclusion boundary $(\zeta-\ell)(\bar{\zeta}-\ell)=1$, are satisfied.

Following are the expressions for stresses in cartesian form in the case of $\delta_3=0$. We use the abbreviations

$$x_1 = x-\frac{a^2}{\ell}, \qquad x_2 = x-\ell, \qquad r_1^2 = x_1^2+y^2,$$
$$r_2^2 = x_2^2+y^2, \quad r^2 = x^2+y^2;$$

$$(\sigma_x)_i = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{1+\frac{1}{r_1^2}(x_1^2-y^2)\left(\frac{2a^2}{\ell^2}+1\right)$$
$$-\frac{2a^2}{\ell^3 r_1^6}(\ell^2-a^2)(x_1^3-3x_1y^2)+\frac{2a^2}{\ell^2 r_1^6}(x_1^4-6x_1^2y^2+y^4)$$
$$-\frac{1}{r_1^4}(x^2-y^2)\Big\}-\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{1+\frac{x_1^2-y^2}{r_1^4}$$
$$\times\left(\frac{6a^4}{\ell^4}-\frac{4a^2}{\ell^2}+\frac{6a^2}{\ell^4}\right)+\frac{2a^4}{\ell^5 r_1^6}(6+2a^2-3\ell^2)$$
$$\times(x_1^3-3x_1y^2)+(6a^2r^2\ell^2+6r^2\ell^2-4r^2\ell^4$$
$$-9a^2\ell^2-8a^4\ell^2+6a^2\ell^4+6a^4)\frac{a^2}{\ell^6 r_1^8}$$
$$-(r_1^2\ell^2-6r^2+6a^2)(x_1^5-10x_1^3y^2+5x_1y^4)\frac{2a^4}{\ell^5 r_1^{10}}$$
$$-\frac{2a^2}{r^4\ell^2}(x^2-y^2)\Big\},$$

$$
\begin{aligned}
(\sigma_y)_i = {} & (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{-1+\Big(\frac{2a^2}{\ell^2}-1\Big)(x_1^2-y^2)\frac{1}{r_1^2} \\
& + \frac{2a^2}{\ell^3 r_1^6}(\ell^2-a^2)(x_1^3-3x_1y^2) + \frac{x^2-y^2}{r^4} \\
& - \frac{2a^2}{\ell^2 r_1^6}(x_1^4-6x_1^2y^2+y^4)\Big\} - \frac{\mu}{\alpha+1}(\delta_1-\delta_2) \\
& \times\Big\{-1+(x_1^2-y^2)\Big(\frac{6a^4}{\ell^4}-\frac{4a^2}{\ell^2}+\frac{6a^2}{\ell^4}\Big)\frac{1}{r_1^4} \\
& + (6+2a^2-\ell^2)(x_1^3-3x_1y^2)\frac{2a^4}{\ell^5 r_1^6} \\
& + (4\ell^4r^2-6a^2r^2\ell^2-6r^2\ell^2+9a^2\ell^2+8a^4\ell^2 \\
& - 6a^2\ell^4+6a^4)(x_1^4-6x_1^2y^2+y^4)\frac{a^2}{\ell^6 r_1^8} \\
& + (\ell^2 r_1^2-6r^2+6a^2)(x_1^5-10x_1^3y^2+5x_1y^4)\frac{2a^4}{\ell^5 r_1^{10}} \\
& + (x^2-y^2)\frac{2a^2}{\ell^2 r^4}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\tau_{xy})_i = {} & 2(\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\,y\Big\{(\ell^2-a^2)(y-3x_1^2)\frac{a^2}{\ell^3 r_1^6} \\
& + (x_1^3-x_1y^2)\frac{4a^2}{\ell^2 r_1^6} + \frac{x_1}{r_1^4} - \frac{x}{r^4}\Big\} \\
& - \frac{2\mu}{\alpha+1}(\delta_1-\delta_2)\,y\Big\{(y^2-3x_1^2)\frac{a^4}{\ell^3 r_1^6} \\
& + (6a^2\ell^2r^2+6r^2\ell^2-4r^2\ell^4-9a^2\ell^2 \\
& - 8a^4\ell^2+6a^2\ell^4)(x_1^3-x_1y^2)\frac{2a^2}{\ell^6 r_1^8} \\
& - (\ell^2 r_1^2-6r^2-12a^2)(5x_1^4-10x_1^2y^2+y^4) \\
& \times\frac{a^4}{\ell^5 r_1^{10}} - \frac{2a^2x}{\ell^2 r^4}\Big\},
\end{aligned}
$$

$$(\sigma_x)_m = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{(2a^2+\ell^2)(x_1^2-y^2)\frac{1}{\ell^2 r_1^4}$$
$$-(\ell^2-a^2)(x_1^3-3x_1y^2)\frac{2a^2}{\ell^3 r_1^6} + (x_1^4-6x_1^2y^2+y^4)\frac{2a^2}{\ell^2 r_1^6}$$
$$-(x_2^2-y^2)\frac{1}{r_2^4} - (x^2-y^2)\frac{1}{r_1^4}\Big\}$$
$$+\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{2(x_2^2-y^2)\frac{1}{r_2^2} + (3a^2-2\ell^2+3)$$
$$\times(x_1^2-y^2)\frac{2a^2}{\ell^4 r_1^4} + (6+2a^2-3\ell^2)(x_1^3-3x_1y^2)\frac{2a^4}{\ell^5 r_1^6}$$
$$+(6a^4+6a^2r^2\ell^2+6r^2\ell^2-4r^2\ell^4-9a^2\ell^2$$
$$-8a^4\ell^2+6a^2\ell^4)\times(x_1^4-6x_1^2y^2+y^4)\frac{a^2}{\ell^6 r_1^8}$$
$$-(\ell^2r_1^2-6r^2+6a^2)(x_1^5-10x_1^3+y^4x_1)\frac{2a^4}{\ell^5 r_1^{10}}$$
$$-\frac{2\ell}{r_2^6}(x_2^3-3x_2y^2)-(x_2^4-6x_2^2y^2+y^4)\frac{3}{r_2^8}$$
$$-\frac{2a^2}{\ell^4 r^4}(x^2-y^2)\Big\},$$

$$(\sigma_y)_m = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{(2a^2-\ell^2)(x_1^2-y^2)\frac{1}{r_1^4\ell^2}$$
$$+(\ell^2-a^2)(x_1^3-3x_1y^2)\frac{a^2}{\ell^2 r_1^6} - (x_1^4-6x_1^2y^2+y^4)\frac{2a^2}{\ell^2 r_1^6}$$
$$+(x_2^2-y^2)\frac{1}{r_2^4} - (x^2-y^2)\frac{1}{r^4}\Big\}$$
$$-\frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{2(x_2^2-y^2)\frac{1}{r_2^4} + (3a^2-2\ell^2+3)$$
$$\times(x_1^2-y^2)\frac{2a^2}{\ell^4 r_1^4} + (6+2a^2-\ell^2)(x_1^3-3x_1y^2)\frac{a^4}{\ell^5 r_1^6}$$
$$+(6a^4-6a^2r^2\ell^2-6r^2\ell^2+4r^2\ell^4+9a^2\ell^2$$
$$+8a^4\ell^2-6a^2\ell^4)(x_1^5-10x_1^3y^2+5x_1y^4)\frac{2a^4}{\ell^5 r_1^{10}}$$
$$+(x_2^2-3x_2y^2)\frac{2\ell}{r_2^6} + (x_2^2-6x_2^2y^2+y^4)\frac{3}{r_2^8}$$
$$+2(x^2-y^2)\frac{a^2}{\ell^2 r^4}\Big\},$$

$$(T_{xy})_m = 2(\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\, y\Big\{ (\ell^2-a^2)(y^2-3x_1^2)\frac{a^2}{\ell^2 r_1^6}$$
$$+\frac{4a^2}{\ell^2 r_1^6}(x_1^3-y^2x_1) - \frac{x_2}{r_2^4} + \frac{x_1}{r_1^4} - \frac{x}{r^4}\Big\}$$
$$-\frac{2\mu(\delta_1-\delta_2)}{\alpha+1}\, y\Big\{\frac{a^4}{\ell^6 r_1^8}(a^4-3x_1^2)+2(6a^2\ell^2 r^2$$
$$+6r^2\ell^2-4r^2\ell^4-9a^2\ell^2-8a^4\ell^2+6a^2\ell^2)$$
$$\times(x_1^3-x_1y^2)\frac{a^2}{\ell^6 r_1^8} - (r^2\ell^2-6r^2+6a^2)$$
$$\times(5x_1^4-10x_1^2y^2+y^4)\frac{1}{r_1^{10}} + \frac{6}{r_2^2}(y^2-3x_2^2)$$
$$-\frac{6}{r_2^2}(x_2^3-y^2x_2) - \frac{2xa^2}{\ell^2 r^4}\Big\}.$$

Hoop stress at the free boundary of the hole is of particular interest. Since normal stress is zero on the boundary, hoop stress may be found from the formula

$$\sigma_\theta\Big|_{z=\sigma} = 4\,\mathrm{Re}\,\phi'(z)$$

Putting

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2} \text{ and } \quad z = r e^{i\theta},$$

we have

$$\sigma_\theta\Big|_{z=\sigma} = 4(\delta_1+\delta_2)(\lambda+\mu)\frac{\alpha-1}{\alpha+1}\frac{a^2}{\ell^2 r_1^2}\cos 2\theta_1$$
$$-\frac{4\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{\frac{1}{\ell^2} + \frac{2a^2}{\ell^3 r_1}\cos\theta_1$$
$$+(4a^2-2\ell^2+3)\frac{a^2}{\ell^4 r_1^2}\cos 2\theta_1$$
$$+(a^2-\ell^2+3)\frac{2a^4}{\ell^5 r_1^3}\cos 3\theta_1 + 3\frac{a^6}{\ell^6 r_1^4}\cos 4\theta_1\Big\},$$

As regards normal and tangential stresses continuously transmitted across $(\zeta-\ell)(\bar{\zeta}-\ell)=1$ by the bond, They are given by

$$
\begin{aligned}
(\sigma_n)\Big|_{z=\zeta} = {} & \frac{(\lambda+\mu)(\delta_1+\delta_2)(\alpha-1)}{\alpha+1}\Big\{-1+\frac{2a^2}{\ell^2 r_1^2}\cos 2\theta_1 \\
& + \frac{2a^2 r}{\ell^2 r_1^3}\cos(2\theta_2-3\theta_1-\theta) - \frac{2a^2}{\ell r_1^3}\cos(2\theta_2-3\theta_1) \\
& + \frac{1}{r_1^2}\cos(2\theta_2-2\theta_1) - \frac{1}{r^2}\cos(2\theta_2-2\theta)\Big\} \\
& - \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{\cos 2\theta_2 + \frac{6a^6}{\ell^6 r_1^4}\cos 4\theta_1 + 4(a^2-\ell^2+3) \\
& \times \frac{a^4}{\ell^5 r_1^3}\cos(3\theta_1) + 2(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 r_1^2}\cos 2\theta_1 \\
& - \frac{2a^2}{\ell^2 r^2}\cos(2\theta_2-2\theta) - \frac{2a^4}{\ell^3 r_1^3}\cos(2\theta_2-3\theta_1) \\
& - \frac{12a^6}{\ell^5 r_1^5}\cos(2\theta_2-5\theta_1) + \frac{12a^6 r}{\ell^6 r_1^5}\cos(2\theta_2-5\theta_1-\theta) \\
& + 2(3a^2-2\ell^2+3)\frac{a^2 r}{\ell^4 r_1^3}\cos(2\theta_2-3\theta_1-\theta) \\
& + 6(a^2-\ell^2+3)\frac{a^4 r}{\ell^5 r_1^4}\cos(2\theta_2-4\theta_1-\theta) \\
& - 3(2a^2-2\ell^2+3)\frac{a^4}{\ell^4 r_1^4}\cos(2\theta_2-4\theta_1)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\tau_{ns})\Big|_{z=\zeta} = {} & (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{\frac{2a^2}{\ell r_1^3}\sin(2\theta_2-3\theta_1) \\
& - \frac{2a^2 r}{\ell^2 r_1^3}\sin(2\theta_2-3\theta_1-\theta) - \frac{1}{r_1^2}\sin(2\theta_2-2\theta_1) \\
& - \frac{1}{r^2}\sin(2\theta_2-2\theta)\Big\} + \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{\sin 2\theta_2 \\
& - \frac{2a^2}{\ell^2 r^2}\sin(2\theta_2-2\theta) - \frac{2a^4}{\ell^3 r_1^3}\sin(2\theta_2-3\theta_1) \\
& - 3(2a^2-2\ell^2+3)\frac{a^4}{\ell^4 r_1^4}\sin(2\theta_2-4\theta_1) \\
& + \frac{12a^6 r}{\ell^6 r_1^5}\sin(2\theta_2-5\theta_1-\theta)
\end{aligned}
$$

$$- 12\frac{a^6}{\ell^5 r_1^5} \sin(2\theta_2 - 5\theta_1)$$
$$+ 2(3a^2 - 2\ell^2 + 3)\frac{a^2 r}{\ell^4 r_1^3} \sin(2\theta_2 - 3\theta_1 - \theta)$$
$$+ 6(a^2 - \ell^2 + 3)\frac{a^4 r}{\ell^5 r_1^4} \sin(2\theta_2 - 4\theta_1 - \theta)\Big\}$$

The hoop stress in the inclusion on its boundary is

$$(\sigma_s)_i\Big|_{z=\zeta} = (\lambda+\mu)(\delta_1+\delta_2)\frac{(\alpha-1)}{\alpha+1}\Big\{-1 + \frac{2a^2}{\ell^2 r_1^2}\cos 2\theta_1$$
$$- \frac{2a^2 r}{\ell^2 r_1^3}\cos(2\theta_2 - 3\theta_1 - \theta) - \frac{1}{r_1^2}\cos(2\theta_2 - 2\theta_1)$$
$$+ \frac{2a^2}{\ell r_1^3}\cos(2\theta_2 - 3\theta_1) + \frac{1}{r^2}\cos(2\theta_2 - 2\theta)\Big\}$$
$$- \frac{\mu(\delta_1-\delta_2)}{\alpha+1}\Big\{-\cos 2\theta_2 + 6\frac{a^4}{\ell^6 r_1^4}\cos 4\theta_1$$
$$+ 4(a^2 - \ell^2 + 3)\frac{a^4}{\ell^5 r_1^3}\cos 3\theta_1 + 2(3a^2 - 2\ell^2 + 3)$$
$$\times \frac{a^2}{\ell^4 r_1^2}\cos 2\theta_1 + \frac{2a^2}{\ell^2 r^2}\cos(2\theta_2 - 2\theta)$$
$$+ \frac{2a^4}{\ell^3 r_1^3}\cos(2\theta_2 - 3\theta_1) + 12\frac{a^6}{\ell^5 r_1^5}\cos(2\theta_2 - 5\theta_1)$$
$$- 12\frac{a^6 r}{\ell^6 r_1^5}\cos(2\theta_2 - 5\theta_1 - \theta) - 2(a^2 - \ell^2 + 3)$$
$$\times \frac{a^4 r}{\ell^5 r_1^5}\cos(2\theta_2 - 4\theta_1 - \theta)$$
$$+ \frac{3a^4}{\ell^4 r_1^4}(2a^2 - 2\ell^2 + 3)\cos(2\theta_2 - 4\theta_1)$$
$$- 6(a^2 - \ell^2 + 3)\frac{a^4 r}{\ell^5 r_1^4}\cos(2\theta_2 - 4\theta_1 - \theta)\Big\}.$$

The value of $(\sigma_s)_m\big|_{z=\zeta}$ can be found easily. It may be seen that the jump in the hoop stress across the boundary is given by

$$(\sigma_s)_i\big|_{z=\zeta} - (\sigma_s)_m\big|_{z=\zeta} = -2(\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}$$
$$+\frac{4\mu(\delta_1-\delta_2)}{\alpha+1}\cos 2\theta_2 .$$

Integration of (90), (91), (92) and (93) with respect to z leads to the following functions

$$\phi_i(z) = \frac{(\lambda+\mu)(\delta_1+\delta_2)}{\alpha+1} z_2 - (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\frac{a^2}{\ell^2 z_1}$$
$$+\frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\left\{\frac{a^6}{\ell^6 z_1^3}\right.$$
$$\left.+\frac{(a^2-\ell^2+3)a^4}{\ell^5 z_1^2}+(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1}\right\}, \tag{94}$$

$$\psi_i(z) = -\frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\left\{\frac{a^2}{\ell z_1^2}-\frac{1}{z_1}+\frac{1}{z}\right\}$$
$$-\frac{(\lambda+\mu)(\delta_1+\delta_2)\ell}{\alpha+1}+\frac{\mu(\delta_1-\delta_2-2i\delta_3)}{\alpha+1}$$
$$\times\left\{-\alpha z_2+\frac{a^4}{\ell^3 z_1^2}+(2a^2-2\ell^2+3)\frac{a^4}{\ell^4 z_1^3}\right.$$
$$\left.+\frac{3a^6}{\ell^5 z_1^2}+\frac{a^2}{\ell^2 z}\right\}+\frac{\mu(\delta_1-\delta_2+2i\delta_3)}{\alpha+1}\frac{a^2}{\ell^2 z}, \tag{95}$$

$$\Phi_m(z) = -\frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_1^2} + \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)$$
$$\times\left\{\frac{a^2}{\ell^4 z_1^2}(3a^2-2\ell^2+3) + \frac{a^4}{\ell^5 z_1^2}(a^2-\ell^2+3) + \frac{a^6}{\ell^6 z_1^3}\right\}$$
$$+ \frac{\mu}{\alpha+1}(\delta_1-\delta_2+2i\delta_3)\frac{1}{z_2}, \tag{96}$$

$$\Psi_m(z) = -\frac{\alpha-1}{\alpha+1}(\lambda+\mu)(\delta_1+\delta_2)\left(\frac{a^2}{\ell z_1^2} - \frac{1}{z_1} + \frac{1}{z} + \frac{r}{z_2}\right)$$
$$+ \frac{\mu}{\alpha+1}(\delta_1-\delta_2-2i\delta_3)\left\{\frac{a^4}{\ell^6 z_1^2} + (2a^2-2\ell^2+3)\frac{a^4}{\ell^4 z_1^3}\right.$$
$$\left. + \frac{3a^6}{\ell^5 z_1^4} + \frac{a^2}{\ell^2 z}\right\} + \frac{\mu}{\alpha+1}(\delta_1-\delta_2+2i\delta_3)$$
$$\times\left(\frac{a^2}{\ell^2 z} + \frac{\ell}{z_2^2} + \frac{1}{z_2^3}\right). \tag{97}$$

The displacement fields can now be obtained from these functions and the equation (2). In cartesian form the displacements are given below for the case when $\delta_3 = 0$.

$$(2\mu u)_i = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\left\{x_2 + \frac{x}{r^2} - \frac{x}{r_1^2}\right.$$
$$- \frac{\alpha a^2 x_1}{\ell^2 r_1^2} + \frac{a^2}{\ell^3 r_1^4}(\ell^2-a^2)(x_1^2-y^2)$$
$$\left. - \frac{a^2}{\ell^2 r_1^4}(x_1^3-3x_1y^2)\right\} + \frac{\mu\alpha}{\alpha+1}(\delta_1-\delta_2)$$
$$\times\left\{x_2 + (3a^2-2\ell^2+3)\frac{a^2 x_1}{\ell^4 r_1^2}\right.$$
$$\left. + \frac{a^6}{\ell^6 r_1^6}(x_1^3-3x_1y^2) + (3+a^2-\ell^2)\frac{a^4}{\ell^5 r_1^4}(x_1^2-y^2)\right\} +$$

$$+ \frac{\mu(\delta_1-\delta_2)}{\alpha+1} \Big\{ 3(a^2-\ell^2+1)\frac{a^4}{\ell^5 r_1^4}(x_1^2-y^2)$$

$$+ 3a^2\ell^2 r_1^2 - 2r_1^2\ell^4 + 3\ell^2 r_1^2 + 2a^6 - 4\ell^2 a^4 + 6a^4$$

$$- 3a^2\ell^2 + 2a^2\ell^4)\ \frac{a^2}{\ell^6 r_1^6}(x_1^3 - 3x_1y^2)$$

$$+ \frac{a^4}{\ell^7 r_1^8}(2a^2\ell^2 r_1^2 - 2\ell^4 r_1^2 + 6\ell^2 r_1^2 + 3a^4 - 3a^2\ell^2)$$

$$\times (x_1^4 - 6x_1^2y^2 + y^4) + \frac{3a^6}{\ell^6 r_1^8}(x_1^5 - 10x_1^3 + 5x_1y^4)$$

$$- \frac{2a^2x}{\ell^2 r^2}\Big\}$$

$$(2\mu v)_i = (\lambda+\mu)(\delta_1+\delta_2)\left(\frac{\alpha-1}{\alpha+1}\right) y \Big\{ 1 + \frac{1}{r^2} - \frac{1}{r_1^2}$$

$$+ \frac{\alpha a^2}{\ell^2 r_1^2} + \frac{2a^2}{\ell^3 r_1^4}(\ell^2-a^2)x_1 - \frac{a^2}{\ell^2 r_1^4}(3x_1^2-y^2)\Big\}$$

$$- \frac{\alpha\mu}{\alpha+1}(\delta_1-\delta_2)\,y\Big\{ 1 + (3a^2-2\ell^2+3)\frac{a^2}{\ell^4 r_1^2}$$

$$+ \frac{2a^4x_1}{\ell^5 r_1^4}(a^2+3-\ell^2) + \frac{a^6}{\ell^6 r_1^6}(3x_1^2-y^2)\Big\}$$

$$+ \frac{\mu}{\alpha+1}(\delta_1-\delta_2)\Big\{ \frac{6a^4x_1}{\ell^5 r_1}(a^2-\ell^2+1)$$

$$+ \frac{a^2}{\ell^6 r_1^6}(3a^2\ell^2 r_1^2 - 2r_1^2\ell^4 + 3\ell^2 r_1^2 + 2a^6$$

$$- 4\ell^2a^4 + 6a^4 - 3a^2\ell^2 + 2\ell^4a^2)(3x_1^2+y^2)$$

$$+ \frac{4a^4}{\ell^7 r_1^8}(2a^2\ell^2 r_1^2 - 2\ell^4 r_1^2 + 6\ell^2 r_1^2$$

$$+ 3a^4 - 3a^2\ell^2)(x_1^3 - y^2x_1) - \frac{2a^2}{\ell^2 r^2}$$

$$+ \frac{3a^6}{\ell^6 r_1^8}(5x_1^4 - 10y^2x_1^2 + y^4) \Big\},$$

$$(2\mu u)_m = (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\Big\{\frac{x}{r^2} - \frac{x_1}{r_1^2}$$

$$+ \frac{x_2}{r_2^2} + (\ell^2-a^2)(x_1^2-y^2)\frac{a^2}{\ell^3 r_1^4} - \frac{\alpha a^2 x_1}{\ell^2 r_1^2}$$

$$- \frac{a^2}{\ell^2 r_1^4}(x_1^3 - 3x_1^2)\Big\}$$

$$+ \frac{\alpha\mu}{\alpha+1}(\delta_1-\delta_2)\Big\{\frac{x_2}{r_2^2} + (3a^2-2\ell^2+3)\frac{a^2x_1}{\ell^2 r_1^2}$$

$$+ (3+a^2-\ell^2)(x_1^2-y^2)\frac{a^4x_1}{\ell^5 r_1^4} + \frac{a^6}{\ell^6 r_1^2}(x_1^3-3x_1y^2)\Big\}$$

$$+ \frac{\mu}{\alpha+1}(\delta_1-\delta_2)\Big\{(r_2^2-1)(x_2^3-3x_2y^2)\frac{1}{r_2^2}$$

$$- \frac{2a^2}{\ell^2 r^2} + \frac{3a^4}{\ell^5 r_1^4}(a^2-\ell^2+1)(x_1^2-y^2)$$

$$+ (3a^2\ell^2r_1^2 - 2\ell^4r_1^2 + 3\ell^2r_1^2 + 2a^6 - 4\ell^2a^4$$

$$+ 6a^4 - 3a^2\ell^2 + 2\ell^4a^2)\frac{a^2}{\ell^6 r_1^6}(x_1^3 - 3x_1y^2)$$

$$+ (2a^2\ell^2r_1^2 + 6\ell^2r_1^2 + 3a^4 - 3a^2\ell^2)\frac{a^4}{\ell^7 r_1^8}$$

$$\times (x_1^4 - 6x_1^2y^2 + y^4) + (x_1^5 - 10x_1^3y^2 + 5y^4x_1)\frac{3a^6}{\ell^7 r_1^8}\Big\},$$

$$
\begin{aligned}
(\lambda \cdot u \cdot v)_m = {} & (\lambda+\mu)(\delta_1+\delta_2)\frac{\alpha-1}{\alpha+1}\, y \Big\{ \frac{1}{r^2} + \frac{1}{r_2^2} - \frac{1}{r_1^2} \\
& + \frac{\alpha a^2}{\ell^2 r_1^2} + (\ell^2-a^2)\frac{2a^2}{\ell^3 r_1^4}\, x_1 - (3x_1^2-y^2)\frac{a^2}{\ell^2 r_1^4} \Big\} \\
& - \frac{\alpha\mu}{\alpha+1}(\delta_1-\delta_2)\, y \Big\{ 1 + (3a^2-2\ell^2+3)\frac{a^2}{\ell^4 r_1^2} \\
& + \frac{2a^4 x_1}{\ell^5 r_1^4}(3+a^2-\ell^2) + \frac{a^6}{\ell^6 r_1^6}(3x_1^2-y^2) \Big\} \\
& + \frac{\mu(\delta_1-\delta_2)}{\alpha+1} \Big\{ \frac{6a^4 x_1}{\ell^5 r_1^4}(a^2-\ell^2+1) + \frac{a^2}{\ell^6 r_1^6} \\
& \times (3a^2\ell^2 r_1^2 - 2r_1^2\ell^2 + 3\ell^2 r_1^2 + 2a^6 - 4a^4\ell^2 \\
& + 6a^4 - 3a^2\ell^2 + 2\ell^4 a^2)(3x_1^2-y^2) \\
& + \frac{4a^4}{\ell^7 r_1^8}(2a^2\ell^2 r_1^2 - 2\ell^4 r_1^2 + 6\ell^2 r_1^2 \\
& + 3a^4 - 3a^2\ell^2)(x_1^3 - y^2 x_1) \\
& + \frac{3a^6}{\ell^6 r_1^8}(5x_1^4 - 10x_1^2 y^2 + y^4) \\
& - \frac{2a^2}{\ell^2 r^2} \Big\}.
\end{aligned}
$$

CHAPTER X

INHOMOGENEITY AND A POINT-FORCE IN THE MATRIX

In this chapter we shall obtain the complex functions for the problem of a point-force acting at a point of the elastic medium which contains a circular inhomogeneity, the point of application of the force is of course outside the inhomogeneity. In chapter VI the concentrated force was applied to a point of the boundary of the inhomogeneity. Here that restriction is not there. The procedure for obtaining the potential functions is similar.

Let ζ be the point of application of the force P. $|\zeta| = b > a$, where a is the radius of the inhomogeneity whose equation in the z - plane is $z\bar{z} \leq a^2$. Let us assume the complex functions in the following form :

$$\phi_m(z) = \frac{P}{2\pi(\alpha_m+1)}\left\{-\log(z-\xi) + A\left(z-\frac{a^2}{\bar{\xi}}\right) + B\left(z-\frac{a^2}{\bar{\xi}}\right)^{-1} + C\log z\right\}, \tag{98}$$

$$\begin{aligned}\psi_m(z) = \frac{\bar{P}}{2\pi(\alpha_m+1)}\Big\{&\alpha_m \log(z-\xi) + \bar{\xi}\frac{P}{\bar{P}}(z-\xi)^{-1} \\ &+ D\log\left(z-\frac{a^2}{\bar{\xi}}\right) + E\left(z-\frac{a^2}{\bar{\xi}}\right)^{-1} \\ &+ F\left(z-\frac{a^2}{\bar{\xi}}\right)^{-2} + G\log z + Hz^{-1} + Iz^{-2}\Big\},\end{aligned} \tag{99}$$

$$\phi_{ih}(z) = \frac{P}{2\pi(\alpha_m+1)}\left\{J\log(z-\xi) + K(z-\xi)^{-1} + Lz + Mz^2\right\}, \tag{100}$$

$$\psi_{ih}(z) = \frac{\bar{P}}{2\pi(\alpha_m+1)}\left\{N\log(z-\xi) + Q(z-\xi)^{-1} + R(z-\xi)^{-2}\right\}, \tag{101}$$

where suffixes ih and m have been used as before to distinguish quantities pertaining to the inhomogeneity and the matrix respectively. The symbols A,B,C, ..., R denote complex constants to be determined from the same boundary conditions listed on page 58 of chapter VI. The continuity of normal and shearing stresses on the boundary means that

$$(\sigma_r - i\tau_{r\theta})_m \big|_{z=\sigma} = (\sigma_r - i\tau_{r\theta})_{ih} \big|_{z=\sigma} . \tag{102}$$

We have, as before, the relation

$$\sigma_r - i\tau_{r\theta} = \phi'(z) + \overline{\phi'(z)} - e^{2i\theta}\left\{\bar{z}\phi''(z) + \psi'(z)\right\}. \tag{103}$$

On substitution from (98) to (101) in (103) and then using (102), we get

$$-P\sigma_1^{-1} - PA\frac{\bar\xi}{\sigma}\sigma_2^{-1} - PB\sigma_2^{-2}\frac{\bar\xi^2}{\sigma^2} + PC\sigma^{-1} - \bar P\sigma_2^{-1}$$

$$-\bar P\bar A\frac{\xi\sigma}{a^2}\sigma_1^{-1} - \bar P\bar B\sigma_1^{-2}\sigma^2\xi^2 + \bar P\bar C\frac{\sigma}{a^2} - P\sigma\sigma_1^{-2}$$

$$+PA\frac{\bar\xi^2}{\sigma}\sigma_2^{-2} + 2PB\frac{\bar\xi^3}{\sigma^2}\sigma_2^{-3} + PC\sigma^{-1}$$

$$-\bar P\alpha_m\frac{\sigma^2}{a^2}\sigma_1^{-1} + P\sigma^2\sigma_1^{-2}\frac{\xi}{a^2} + \bar P D\bar\xi\frac{\sigma}{a^2}\sigma_2^{-1} + \bar P E\frac{\bar\xi^2}{a^2}\sigma_2^{-2}$$

$$-2\bar P F\frac{\bar\xi^3}{\sigma a^2}\sigma_2^{-3} - \bar P G\frac{\sigma}{a^2} + \frac{H\bar P}{a^2} + 2\frac{\bar P I}{\sigma a^2} - PJ\sigma_1^{-1}$$

$$+PK\sigma_1^{-2} + PL - \bar P\bar J\sigma_2^{-1} + \bar P\bar K\sigma_2^{-2} - \bar P\bar L$$

$$-2\bar P\bar M\frac{a^2}{\sigma} - PJ\sigma\sigma_1^{-2} + 2PK\sigma\sigma_1^{-3}$$

$$+\bar P N\frac{\sigma^2}{a^2}\sigma_1^{-1} - PQ\frac{\sigma^2}{a^2}\sigma_1^{-2} - 2\bar P R\frac{\sigma^2}{a^2}\sigma_1^{-3} = 0,$$

where $\sigma_1 = \sigma - \xi$ and $\sigma_2 = \bar\sigma - \bar\xi$. The left-hand side equation can be put in the form of a polynomial in σ . On equating the coefficients of various powers of σ to zero we get a set of nine equations

$$F = \frac{P}{\bar P}\bar\xi B,$$

$$R = \frac{P}{\bar P}\frac{a^2}{\xi}K,$$

$$PC + I\frac{\bar{P}}{a^2} - 2\bar{P}\bar{M}a^2 = 0,$$

$$U - I + \bar{H} - L - \alpha_m = 0,$$

$$\begin{aligned}
&\bar{P}(\bar{J}+1)\bar{\zeta} + (a^2+2b^2)\bar{\zeta}\frac{D\bar{P}}{a^2} \\
&-(2a^2+b^2)\bar{\zeta}\frac{\bar{P}N}{a^2} - \bar{P}\bar{F}\frac{b^4}{a^4} \\
&+\frac{P}{a^2}\bar{\zeta}^3 - PQ\frac{\bar{\zeta}}{a^2} + \bar{P}\bar{K} + \bar{P}\bar{\zeta}^2\frac{E}{a^2} \\
&+\left(H\frac{\bar{P}}{a^2} - PL - \bar{P}\bar{L}\right)\bar{\zeta}^2 \\
&-2(\bar{C}-G)\frac{\bar{P}}{a^2}\bar{\zeta}(a^2+b^2) = 0,
\end{aligned}$$

$$\begin{aligned}
&-2P(J+1)\bar{\zeta}^2 + \bar{P}\bar{A}(2a^2+b^2)\frac{b^2}{a^2} \\
&-(a^2+2b^2)\bar{P}\alpha_m + (a^2+2b^2)\bar{P}N + 2PA\bar{\zeta}^2 \\
&-\bar{P}(\bar{J}+1)(a^2+2b^2) - (2a^2+b^2)D\bar{P}\frac{b^2}{a^2} \\
&-2PK\frac{\bar{\zeta}^2}{\zeta} + 2\bar{P}\cdot\bar{B}\,\zeta\frac{b^2}{a^2} - 2P\bar{\zeta}^2 \\
&+2\bar{P}Q\bar{\zeta} + 2P\frac{B}{a^2}\bar{\zeta}^3 - 2\bar{P}\bar{K}\zeta \\
&-2\bar{P}E\bar{\zeta}\frac{b^2}{a^2} - 2\left(H\frac{\bar{P}}{a^2} - PL - \bar{P}\bar{L}\right)\bar{\zeta}(a^2+2b^2) \\
&+(a^4+b^4+4a^2b^2)\frac{\bar{P}}{a^2}(\bar{C}-G) = 0,
\end{aligned}$$

$$-(a^2+2b^2)\bar{A}\bar{P}\zeta + P(\bar{\zeta}+1)(4a^2+b^2)\bar{\zeta} + a^2\bar{P}\alpha_m\zeta$$
$$-\bar{P}Na^2\zeta - AP\bar{\zeta}(a^2+4b^2) + \bar{P}(\bar{\zeta}+1)\zeta(2a^2+b^2)$$
$$+b^2\zeta D\bar{P} + 4PK\frac{a^2}{\zeta}\bar{\zeta} + P\bar{\zeta}a^2 - \bar{P}\bar{B}\zeta^2 - \bar{P}Qa^2$$
$$+PK\bar{\zeta}^2 - PB\bar{\zeta}^2 - 4PB\frac{b^2}{a^2}\bar{\zeta}^2 + \bar{P}\bar{K}\zeta^2$$
$$+\left(H\frac{\bar{P}}{a^2} - PL - \bar{P}\bar{L}\right)(a^4+b^4+4a^2b^2) + E\bar{P}\frac{b^4}{a^2}$$
$$-2\bar{P}(\bar{C}-G)\zeta(a^2+b^2) \qquad = 0,$$

$$-2(a^2+b^2)a^2P(\zeta+1) + a^2\zeta^2\bar{P}\bar{A}$$
$$+2APb^2(a^2+b^2) - \bar{P}(\bar{\zeta}+1)a^2\zeta^2$$
$$+2PBb^2\bar{\zeta} + 2PB\frac{b^4}{a^2}\bar{\zeta} - 2PK\frac{a^4}{\zeta}$$
$$-2PKa^2\bar{\zeta} - 2\left(\frac{H\bar{P}}{a^2} - PL - \bar{P}\bar{L}\right)a^2\zeta(a^2+b^2)$$
$$+\bar{P}a^2\zeta^2(\bar{C}-G) \qquad = 0,$$

$$P(\zeta+1)a^4\zeta - APb^2a^2\zeta - PBb^4$$
$$+PKa^4 + \left(H\frac{\bar{P}}{a^2} - PL - \bar{P}\bar{L}\right)a^4\zeta^2 \qquad = 0.$$

The condition that displacements are continuous across $\sigma\bar{\sigma} = a^2$ means

$$(u-iv)_m\Big|_{z=\sigma} = (u-iv)_{in}\Big|_{z=\sigma}.$$

In other words

$$\frac{1}{\mu_m}\left\{\alpha_m\overline{\Phi_m(\sigma)}-\bar\sigma\,\phi_m(\sigma)-\psi_m(\sigma)\right\}=\frac{1}{\mu_{ih}}\left\{\alpha_{ih}\overline{\Phi_{ih}(\sigma)}-\bar\sigma\,\phi_{ih}(\sigma)-\psi_{ih}(\sigma)\right\}. \tag{104}$$

On substitution from (98) to (101), the above equation leads to

$$\beta\Big[\alpha_m\bar P\left\{-\log\sigma_2+\bar A\log(\bar\sigma-\tfrac{a^2}{\bar\rho})+\bar B(\bar\sigma-\tfrac{a^2}{\rho})^{-1}+\bar C\log\bar\sigma\right\}$$

$$-P\bar\sigma\left\{-\sigma_1^{-1}+A(\sigma-\tfrac{a^2}{\bar\rho})^{-1}-B(\sigma-\tfrac{a^2}{\bar\rho})^{-2}+C\sigma^{-1}\right\}$$

$$-\bar P\left\{\alpha_m\log\sigma_1+\bar\delta\frac{P}{\bar P}\sigma_1^{-1}+D\log(\sigma-\tfrac{a^2}{\bar\rho})+E(\sigma-\tfrac{a^2}{\bar\rho})^{-1}\right.$$

$$\left.+F(\sigma-\tfrac{a^2}{\bar\rho})^{-2}+G\log\sigma+H\sigma^{-1}+I\sigma^{-2}\right\}\Big]$$

$$=\alpha_{ih}\bar P\left\{\bar J\log\sigma_2+\bar K\sigma_2^{-1}+\bar L\bar\sigma+\bar M\bar\sigma^2\right\}$$

$$-P\bar\sigma\left\{\bar J\sigma_1^{-1}-K\sigma_1^{-2}+L+2M\sigma\right\}$$

$$-\bar P\left\{N\log\sigma_1+Q\sigma_1^{-1}+R\sigma_1^{-2}\right\},$$

where $\beta=\alpha_{ih}/\mu_m$.

The vanishing of the logarithmic terms separately yields a set of three equations

$$\alpha_m\bar A+\alpha_m\bar C+D+G=0,$$

$$-\beta\alpha_m\bar C-\beta G+N+\alpha_{ih}\bar J=0,$$

$$-\beta D-2\beta\alpha_m+\beta\alpha_m\bar A+N-\alpha_{ih}\bar J=0.$$

The other terms (after putting them in the form of a polynomial in σ and equating to zero the coefficients of various powers of σ) yields a set of five equations

$$K - \beta \frac{\alpha_m}{\alpha_{ih}} \frac{b^2}{a^2} B = 0,$$

$$\beta a^2 P C + \beta I P + \alpha_m \bar{P} \bar{M} a^4 = 0,$$

$$P K \frac{a^4}{\rho} + \beta a^4 P + P J a^4 - \beta P A a^2 b^2$$
$$- \beta P B \bar{\rho} b^2 + \left(\beta \frac{\bar{P} H}{a^2} + \alpha_{ih} \bar{P} \bar{L} - P L \right) a^4 \rho = 0,$$

$$- 2 P \beta a^2 \bar{\rho} - a^2 P J \bar{\rho} - P K \frac{a^2}{\rho} \bar{\rho} + \bar{P} Q a^2 + P \beta A a^2 \bar{\rho}$$
$$- \beta P B \bar{\rho} a^2 - \beta \bar{P} E b^2 - \left(\beta \frac{\bar{P} H}{a^2} + \alpha_{ih} \bar{P} \bar{L} - P L \right) (a^2 + b^2) a^2 = 0,$$

$$\beta \bar{P} E \bar{\rho} + \alpha_{ih} \bar{P} \bar{K} \rho + \beta P \bar{\rho}^2 - P Q \bar{\rho}$$
$$- \beta \alpha_m \bar{P} \bar{B} \rho + \left(\beta \frac{\bar{P} H}{a^2} + \alpha_{ih} \bar{P} \bar{L} - P L \right) a^2 \bar{\rho} = 0.$$

The application of the last boundary condition yields two equations

$$A + C = 0,$$

$$D + G = 0.$$

The condition of displacements being single-valued yields one equation

$$\alpha_m \bar{A} + \alpha_m \bar{C} + D + G = 0.$$

The set of twenty equations was found to be consistent. The values of the constants are given below :

$$A = -\nu_1 \alpha_m, \qquad C = \nu_1 \alpha_m,$$

$$B = \frac{\bar{P}}{P} \frac{r^2 - 1}{r 6} \nu_1 \frac{\rho^3}{a^2}, \qquad D = \nu_2,$$

$$C = \frac{P}{\bar{P}} \nu_1 \alpha_m \bar{\rho} - \nu_1 \frac{r^2 - 1}{r 2} \rho, \qquad G = -\nu_2,$$

$$\Psi_m(z) = \frac{\bar{P}}{2\pi(\alpha_m+1)}\Big\{ \alpha_m \log(z-\zeta) + \nu_2 \log\Big(z-\frac{a^2}{\bar{\zeta}}\Big)$$

$$- \nu_1 \frac{r^2-1}{r^2} \zeta \Big(z-\frac{a^2}{\bar{\zeta}}\Big)^{-1} + \frac{r^2-1}{r^4} \nu_1 \zeta^2 \Big(z-\frac{a^2}{\bar{\zeta}}\Big)^{-2}$$

$$- \nu_2 \log z + \nu_1 \frac{r^2-1}{r^2} \zeta z^{-1} + \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1} \frac{\zeta z^{-1}}{r^2}\Big\}$$

$$+ \frac{P}{2\pi(\alpha_m+1)} \Big\{ \bar{\zeta} (z-\zeta)^{-1} + \nu_1 \alpha_m \bar{\zeta} \Big(z-\frac{a^2}{\bar{\zeta}}\Big)^{-1}$$

$$- \nu_1 \alpha_m \bar{\zeta} z^{-1} - \nu_2 \frac{\bar{\zeta} z^{-1}}{r^2} - \nu_1 \alpha_m a^2 z^{-2}$$

$$+ \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1} \frac{\bar{\zeta} z^{-1}}{r^2}\Big\}, \tag{106}$$

$$\Phi_{ih}(z) = \frac{P}{2\pi(\alpha_m+1)} \Big\{ (\nu_2-1) \log(z-\zeta) + \frac{(\nu_2-1)(\beta-1)}{2r^2a^2} \bar{\zeta} z$$

$$\times \Big(\frac{1}{(2\beta+\alpha_{ih}-1)} - \frac{1}{\alpha_{ih}+1}\Big) + \frac{\bar{P}}{2\pi(\alpha_m+1)}$$

$$\times \zeta z \frac{(\nu_2-1)(\beta-1)}{2r^2a^2}\Big(\frac{1}{2\beta+\alpha_{ih}-1} + \frac{1}{\alpha_{ih}+1}\Big), \tag{107}$$

$$\Psi_{ih}(z) = \frac{\bar{P}}{2\pi(\alpha_m+1)} (1-\nu_1)\alpha_m \log(z-\zeta)$$

$$+ \frac{P}{2\pi(\alpha_m+1)} \bar{\zeta}(z-\zeta)^{-1} \frac{(1-\nu_1)r^2+(\nu_1-\nu_2)}{r^2}. \tag{108}$$

The knowledge of these complex potentials enables one to determine the elastic fields everywhere in the matrix and the inhomogeneity.

If there is a continuous distribution of point-forces along any curve not intersecting the boundary of the inhomogeneity, the complex potentials may be obtained by simple integration along the curve. Thus for example, it may be seen that

$$\Phi_m'(z) = \frac{1}{2\pi(\alpha_m+1)}\Big[\int_\gamma \frac{P\,ds}{\zeta - z} - \int_\gamma \nu_1\alpha_m P\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-1}ds - \int_\gamma \bar P\nu_1\zeta^3\frac{(r^2-1)}{a^2r^6}\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-2}ds + \int_\gamma P\nu_1\alpha_m z^{-1}ds\Big], \tag{109}$$

$$\begin{aligned}\Psi_m'(z) = \frac{1}{2\pi(\alpha_m+1)}\Big[&\int_\gamma \bar P\alpha_m(z-\zeta)^{-1}ds + \int_\gamma \bar P\nu_2\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-1}ds \\ &+ \int_\gamma \bar P\nu_1\frac{r^2-1}{r^2}\zeta\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-2}ds - 2\int_\gamma \bar P\nu_1\frac{r^2-1}{r^4}\zeta^2\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-3}ds \\ &- \int_\gamma \nu_2\bar P z^{-1}ds - \int \bar P\nu_1\frac{r^2-1}{r^2}\zeta z^{-2}ds \\ &- \int_\gamma \bar P\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\frac{\zeta z^{-2}}{r^2}ds - \int_\gamma P\bar\zeta(z-\zeta)^{-2}ds \\ &- \int_\gamma \bar P\nu_1\alpha_m\bar\zeta\Big(z-\frac{a^2}{\bar\zeta}\Big)^{-2}ds + \int_\gamma P\bar\zeta\nu_1 z^{-2}\alpha_m\,ds \\ &+ \int_\gamma P\nu_2\bar\zeta\frac{z^{-2}}{r^2}ds - \int_\gamma P\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\frac{\bar\zeta z^{-2}}{r^2}ds \\ &+ 2\int_\gamma P\nu_1\alpha_m a^2 z^{-3}ds\Big],\end{aligned} \tag{110}$$

$$\begin{aligned}\Phi_{ih}'(z) = \frac{1}{2\pi(\alpha_m+1)}\Big[&\int_\gamma P(\nu_2-1)(z-\zeta)^{-1}ds \\ &+ \int_\gamma P\frac{(\nu_2-1)(\beta-1)}{2a^2}\Big(\frac{1}{2\beta+\alpha_{ih}-1} - \frac{1}{\alpha_{ih}+1}\Big)\frac{\bar\zeta\,ds}{r^2} \\ &+ \int_\gamma \frac{\bar P(\nu_2-1)(\beta-1)}{2a^2r^2}\zeta\Big(\frac{1}{2\beta+\alpha_{ih}-1} + \frac{1}{\alpha_{ih}+1}\Big)\Big],\end{aligned} \tag{111}$$

$$\Psi'_{ih}(z) = \frac{1}{2\pi(\alpha_m+1)}\left[\int_\gamma \bar{P}\,(1-\nu_1)\,\alpha_m\,(z-\zeta)^{-1}\,ds \right.$$

$$\left. - \int_\gamma P\left\{\frac{(1-\nu_1)r^2+(\nu_1-\nu_2)}{r^2}\right\}\bar{\zeta}\,(z-\zeta)^{-2}\,ds\right],$$

(112)

where γ is the curve and ds is differential arc length along it. P is the distribution of point-force layer. These results shall be used in the next chapter.

CHAPTER XI

CIRCULAR INCLUSION IN THE PRESENCE OF A CIRCULAR INHOMOGENEITY

In chapter IX the solution was given for a circular inclusion deforming in the presence of a cavity. In this chapter a different model is taken. Here in place of a cavity we have an inhomogeneity with elastic constants different from those of the matrix. Of course, a perfect bond between the different materials is assumed, implying thereby the continuity of the normal and shearing stresses across the common boundary.

We choose the reference frame in accordance with Figure 10, p. 138. The origin lies at the centre of the inhomogeneity of radius a. The centre of the inclusion lies on the x- axis at a

distance ℓ ($\ell > a + 1$) from the origin, 1 being the radius of the inclusion. Thus $z\bar{z} \le a^2$ and $(z-\ell)(\bar{z}-\ell) \le 1$ represent the inhomogeneity and the inclusion regions respectively. The remaining part of the z-plane is the matrix.

The inclusion region $(z-\ell)(\bar{z}-\ell) \le 1$ tends to undergo a displacement characterised by

$$u = \delta_1 (x-\ell) + \delta_3 y,$$
$$v = \delta_3 (x-\ell) + \delta_2 y$$

Therefore the point-force layer obtained from

$$\left.\begin{aligned} \sigma_x^\circ &= -\{\lambda_m(\delta_1+\delta_2) + 2\mu_m \delta_1\}, \\ \sigma_y^\circ &= -\{\lambda_m(\delta_1+\delta_2) + 2\mu_m \delta_2\}, \\ \tau_{xy}^\circ &= -2\mu_m \delta_3, \end{aligned}\right\} \qquad (113)$$

and the formulae (22), are given by

$$\begin{aligned} P\,ds &= -i(\lambda_m+\mu_m)(\delta_1+\delta_2)\,d\zeta + i\mu_m(\delta_1-\delta_2+2i\delta_3)\,d\bar{\zeta}, \\ \bar{P}\,ds &= i(\lambda_m+\mu_m)(\delta_1+\delta_2)\,d\bar{\zeta} - i\mu_m(\delta_1-\delta_2-2i\delta_3)\,d\zeta, \end{aligned} \qquad (114)$$

where λ_m, μ_m are Lame's constants of the material of the inclusion or matrix. It may be emphasised here that the inclusion and the matrix are formed the same elastic material. We substitute (114) in (109) - (112) and evaluate the contour integrals. Note that $r = \frac{|\zeta|}{a}$

The expressions simplify after substituting

$$z_1 = z - a^2/\ell,$$
$$z_2 = z - \ell.$$

The details of integration are again not given here. Henceforth a

third region (the inclusion) comes into picture, suffix i has been attached to quantities pertaining to it. Thus

$$
\begin{aligned}
\phi'_{ih}(z) = {} & \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)(\nu_2-1)\left\{\frac{1}{z_2^2} - \frac{\beta-1}{2\ell^2(2\beta+\alpha_{ih}-1)}\right. \\
& \left. + \frac{\beta-1}{2\ell^2(\alpha_{ih}+1)}\right\} - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
& \times\left\{\frac{(\nu_2-1)(\beta-1)}{2\ell^2(2\beta+\alpha_{ih}-1)} + \frac{(\nu_2-1)(\beta-1)}{2\ell^2(\alpha_{ih}+1)}\right\},
\end{aligned}
\tag{115}
$$

$$
\begin{aligned}
\psi'_{ih}(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(1-\nu_1)(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{1}{z_2^2} \\
& + \frac{\mu_m}{\alpha_m+1}(\nu_1-\nu_2)(\delta_1-\delta_2+2i\delta_3)\left(\frac{a^2}{\ell^2 z_2^2} - \frac{2a^2}{\ell z_2^3}\right) \\
& - \frac{\mu_m}{\alpha_m+1}(1-\nu_1)(\delta_1-\delta_2+2i\delta_3)\left(\frac{2\ell}{z_2^3} + \frac{3}{z_2^4}\right),
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\phi'_m(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\nu_1\frac{a^2}{\ell^2 z_1^2} \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{1}{z_2^2} - \frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
& \times\left\{(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2} + (a^2-\ell^2+3)\frac{2a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4}\right\},
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
\Psi_m'(z) = {} & \frac{\alpha_m - 1}{\alpha_m + 1}(\lambda_m + \mu_m)(\delta_1 + \delta_2)\left(\frac{1}{z_2^2} + \frac{2a^2 \nu_1}{\ell z_1^3} + \frac{\nu_1}{z^2} - \frac{\nu_1}{z_1^2}\right) \\
& - \frac{\mu_m}{\alpha_m + 1}(\delta_1 - \delta_2 - 2i\delta_3)\Big[(\nu_2 - \nu_1)\frac{a^2}{\ell^2 z_1^2} + \frac{2a^4 \nu_1}{\ell^3 z_1^3} + \frac{12 a^6 \nu_1}{\ell^5 z_1^5} \\
& + (2a^2 - 2\ell^2 + 3)\frac{3a^4 \nu_1}{\ell^4 z_1^4} + \left\{\nu_1 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right\}\frac{a^2}{\ell^2 z^2}\Big] \\
& - \frac{\mu_m}{\alpha_m + 1}(\delta_1 - \delta_2 + 2i\delta_3)\Big[\frac{2\ell}{z_2^3} + \frac{3}{z_2^4} \\
& + \left\{\nu_2 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right\}\frac{a^2}{\ell^2 z^2}\Big],
\end{aligned}
\tag{118}
$$

$$
\begin{aligned}
\Phi_i'(z) = {} & \frac{(\lambda_m + \mu_m)(\delta_1 + \delta_2)}{\alpha_m + 1} + \frac{\alpha_m - 1}{\alpha_m + 1}(\delta_1 + \delta_2)(\lambda_m + \mu_m)\frac{a^2 \nu_1}{\ell^2 z_1^2} \\
& - \frac{\nu_1 \mu_m}{\alpha_m + 1}(\delta_1 - \delta_2 - 2i\delta_3)\Big[(3a^2 - 2\ell^2 + 3)\frac{a^2}{\ell^4 z_1^2} \\
& + 2(a^2 - \ell^2 + 3)\frac{a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4}\Big],
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
\Psi_i'(z) = {} & \frac{\alpha_m - 1}{\alpha_m + 1}(\lambda_m + \mu_m)(\delta_1 + \delta_2)\nu_1\left(\frac{2a^2}{\ell z_1^3} + \frac{1}{z^2} - \frac{1}{z_1^2}\right) \\
& - \frac{\mu_m}{\alpha_m + 1}(\delta_1 - \delta_2 - 2i\delta_3)\Big[\alpha_m + (\nu_2 - \nu_1)\frac{a^2}{\ell^2 z_1^2} + \frac{2a^4 \nu_1}{\ell^3 z_1^3} \\
& + \frac{12 a^6 \nu_1}{\ell^5 z_1^5} + (2a^2 - 2\ell^2 + 3)\frac{3a^4 \nu_1}{\ell^4 z_1^4} + \left\{\nu_1 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right\}\frac{a^2}{\ell^2 z^2}\Big] \\
& - \Big[\frac{\mu_m}{\alpha_m + 1}(\delta_1 - \delta_2 + 2i\delta_3)\left\{\nu_2 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1}\right\}\frac{a^2}{\ell^2 z^2}\Big].
\end{aligned}
\tag{120}
$$

The complex functions $\phi_{ih}'(z)$, $\psi_i'(z)$, $\phi_m'(z)$, and $\psi_m'(z)$ with the equation (1) directly give the stress fields in the inhomogeneity and the matrix. But $\phi_i'(z)$ and $\psi_i'(z)$ with (1) give only a part of the stress field in the inclusion. The already existing stress field given by (113) must be added to it. A check on the analysis is provided by verifying that the normal and the shearing stresses are continuous across the boundaries, both of the inclusion and the inhomogeneity. This has been done. Since all the boundary conditions, the conditions of regularity at infinity, equilibrium equations, continuity equations etc. are satisfied, the solution given here is the solution.

The stresses on the edges of the inclusion and the inhomogeneity are of special interest. The explicit expressions for them are given below. Writing

$$z = r e^{i\theta}, \quad z_1 = r_1 e^{i\theta_1} \text{ and } \quad z_2 = r_2 e^{i\theta_2}$$

the normal and tangential stresses along the inclusion boundary are

$$\begin{aligned}(\sigma_n)_{r_2=1} &= \frac{\alpha_m - 1}{\alpha_m + 1}(\lambda_m + \mu_m)(\delta_1 + \delta_2)\Big[-1 + \nu_1\Big\{\frac{2a^2}{\ell^2 r_1^2}\cos 2\theta_1 \\ &+ \frac{1}{r_1^2}\cos(2\theta_2 - 2\theta_1) - \frac{1}{r^2}\cos(2\theta_2 - 2\theta) - \frac{2a^2}{\ell r_1^3}\cos(2\theta_2 - 3\theta_1) \\ &+ \frac{2a^2}{\ell^2 r_1^3} r \cos(2\theta_2 - 3\theta_1 - \theta)\Big\}\Big] - \frac{\mu_m}{\alpha_m + 1}(\delta_1 - \delta_2) \times\end{aligned}$$

$$\times\Big[\cos 2\theta_2 + \frac{6a^6\nu_1}{\ell^4 r_1^4}\cos 4\theta_1 + \frac{4a^4\nu_1}{\ell^5 r_1^3}(a^2-\ell^2+3)\cos 3\theta_1$$

$$+\frac{2a^2\nu_1}{\ell^4 r_1^2}(3a^2-2\ell^2+3)\cos 2\theta_1 - \frac{2a^4\nu_1}{\ell^3 r_1^3}\cos(2\theta_2-3\theta_1)$$

$$-\frac{3a^4\nu_1}{\ell^4 r_1^4}(2a^2-2\ell^2+3)\cos(2\theta_2-4\theta_1) - \frac{12a^6\nu_1}{\ell^5 r_1^5}\cos(2\theta_2-5\theta_1)$$

$$-(\nu_2-\nu_1)\frac{a^2}{\ell^2 r_1^2}\cos(2\theta_2-2\theta_1) - (\nu_1+\nu_2)\frac{a^2}{\ell^2 r^2}\cos(2\theta_2-2\theta)$$

$$+\frac{2(\beta-1)(\nu_2-1)}{2\beta+\alpha_{in}-1}\,\frac{a^2}{\ell^2 r^2}\cos(2\theta_2-2\theta_1) + \frac{2a^2 r\nu_1}{\ell^4 r_1^3}(3a^2-2\ell^2+3)$$

$$\times\cos(2\theta_2-\theta-3\theta_1) + \frac{6a^4\nu_1 r}{\ell^5 r_1^4}(a^2-\ell^2+3)\cos(2\theta_2-\theta-4\theta_1)$$

$$+\frac{12a^6\nu_1 r}{\ell^6 r_1^5}\cos(2\theta_2-\theta-5\theta_1)\Big] + \frac{2\mu_m\delta_3}{\alpha_m+1}\Big[-\sin 2\theta_2$$

$$+\frac{6a^4\nu_1}{\ell^3 r_1^3}\sin 4\theta_1 + \frac{4a^4\nu_1}{\ell^5 r_1^3}(a^2-\ell^2+3)\sin 3\theta_1 + \frac{2a^2\nu_1}{\ell^4 r_1^2}$$

$$\times(3a^2-2\ell^2+3)\sin 2\theta_1 + \frac{2a^4\nu_1}{\ell^3 r_1^3}\sin(2\theta_2-3\theta_1) + \frac{3\nu_1 a^4}{\ell^4 r_1^4}$$

$$\times(2a^2-2\ell^2+3)\sin(2\theta_2-4\theta_1) + \frac{12a^6\nu_1}{\ell^5 r_1^5}\sin(2\theta_2-5\theta_1)$$

$$+(\nu_2-\nu_1)\frac{a^2}{\ell^2 r_1^2}\sin(2\theta_2-2\theta_1) + (\nu_1-\nu_2)\frac{a^2}{\ell^2 r^2}\sin(2\theta_2-2\theta)$$

$$- \frac{2a^2 \nu_1 r}{\ell^4 r_1^3} (3a^2+3-2\ell^2) \sin(2\theta_2-\theta-3\theta_1) - \frac{6a^4 \nu_1 r}{\ell^5 r_1^4} (a^2-\ell^2+3)$$

$$\times \sin(2\theta_2-4\theta_1-\theta) - \frac{12a^6 \nu_1 r}{\ell^6 r_1^5} \sin(2\theta_2-\theta-5\theta_1) \Big],$$

$$(T_{rs})_{r_2=1} = (\lambda_m+\mu_m)(\delta_1+\delta_2) \frac{\alpha_m-1}{\alpha_m+1} \Big[- \frac{2\nu_1 a^2 r}{\ell^2 r_1^3} \sin(2\theta_2-\theta-3\theta_1)$$

$$+ \frac{2a^2 \nu_1}{\ell r_1^3} \sin(2\theta_2-3\theta_1) + \frac{\nu_1}{r^2} \sin(2\theta_2-2\theta)$$

$$- \frac{\nu_1}{r_1^2} \sin(2\theta_2-2\theta_1) \Big] + \frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1} \Big[\sin 2\theta_2$$

$$-(\nu_2-\nu_1) \frac{a^2}{\ell^2 r_1^2} \sin(2\theta_2-2\theta_1) - \frac{a^2}{\ell^2 r^2} (\nu_1+\nu_2) \sin(2\theta_2-2\theta)$$

$$+ \frac{2(\nu_2-1)(\beta-1)a^2}{(2\beta+\alpha_{ch}-1)\ell^2 r^2} \sin(2\theta_2-2\theta)$$

$$- \frac{2\nu_1 a^4}{\ell^3 r_1^3} \sin(2\theta_2-3\theta_1) - \frac{3a^4 \nu_1}{\ell^4 r_1^4} (2a^2-2\ell^2+3)$$

$$\times \sin(2\theta_2-4\theta_1) - \frac{12a^6 \nu_1}{\ell^5 r_1^5} \sin(2\theta_2-5\theta_1)$$

$$+ \frac{2a^2 r \nu_1}{\ell^4 r_1^3} (3a^2+3-2\ell^2) \sin(2\theta_2-\theta-3\theta_1)$$

$$+ \frac{6a^4 r \nu_1}{\ell^5 r_1^4} (a^2-\ell^2+3) \sin(2\theta_2-4\theta_1-\theta) + \frac{12a^6 \nu_1 r}{\ell^6 r_1^5}$$

$$\sin(2\theta_2-\theta-5\theta_1) \Big] + \frac{2\mu_m\delta_3}{\alpha_m+1} \Big[-\cos 2\theta_2 + (\nu_2-\nu_1) \frac{a^2}{\ell^2 r_1^2}$$

$$\times \cos(2\theta_2-2\theta_1) - (\nu_1-\nu_2) \frac{a^2}{\ell^2 r^2} \cos(2\theta_2-2\theta)$$

$$+ \frac{2a^4 \nu_1}{\ell^3 r_1^3} \cos(2\theta_2-3\theta_1) + \frac{3a^4 \nu_1}{\ell^4 r_1^4} (2a^2-2\ell^2+3) \cos(2\theta_2-4\theta_1)$$

$$+12\frac{a^6\nu_1}{\ell^5 r_1^5}\cos(2\theta_2-5\theta_1)+\frac{2a^2\nu_1 r}{\ell^4 r_1^3}(3a^2-2\ell^2+3)$$
$$\times\cos(2\theta_2-\theta-3\theta_1)-\frac{6a^4 r\nu_1}{\ell^5 r_1^4}(a^2-\ell^2+3)$$
$$\times\cos(2\theta_2-\theta-4\theta_1)-12\frac{a^6\nu_1 r}{\ell^6 r_1^5}\cos(2\theta_2-\theta-5\theta_1)$$

The hoop stress on the inclusion boundary is given by

$$\left[(\sigma_s)_{r_2=1}\right]_i=(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{\alpha_m-1}{\alpha_m+1}\left[-1+\frac{2a^2\nu_1}{\ell^2 r_1^2}\cos 2\theta_1\right.$$
$$-\frac{\nu_1}{r_1^2}\cos(2\theta_2-2\theta_1)+\frac{\nu_1}{r^2}\cos(2\theta_2-2\theta)$$
$$\left.+\frac{2a^2\nu_1}{\ell r_1^3}\cos(2\theta_2-3\theta_1)-\frac{2a^2 r\nu_1}{\ell^2 r_1^3}\cos(2\theta_2-\theta-3\theta_1)\right]$$
$$-\frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1}\left[-\cos 2\theta_2+\frac{6a^6\nu_1}{\ell^6 r_1^4}\cos 4\theta_1\right.$$
$$+\frac{4a^4\nu_1}{\ell^5 r_1^3}(a^2-\ell^2+3)\cos 3\theta_1+\frac{2a^2\nu_1}{\ell^4 r_1^2}$$
$$\times(3a^2-2\ell^2+3)\cos 2\theta_1+\frac{2a^4\nu_1}{\ell^3 r_1^3}$$
$$\times\cos(2\theta_2-3\theta_1)+\frac{3\nu_1 a^4}{\ell^4 r_1^4}(2a^2-2\ell^2+3)$$
$$\times\cos(2\theta_2-4\theta_1)+12\frac{a^6\nu_1}{\ell^5 r_1^5}\cos(2\theta_2-5\theta_1)$$
$$+(\nu_2-\nu_1)\frac{a^2}{\ell^2 r_1^2}\cos(2\theta_2-2\theta_1)$$

$$+\left\{ v_1 + v_2 - \frac{2(v_2-1)(\beta-1)}{2\beta+\alpha_{ca}-1} \right\} \frac{a^2}{\ell^2 r^2} \cos(2\theta_2 - 2\theta)$$

$$- \frac{2a^2 r v_1}{\ell^4 r_1^3} (3a^2 - 2\ell^2 + 3) \cos(2\theta_2 - \theta - 3\theta_1) - \frac{6a^4 v_1 r}{\ell^5 r_1^4}$$

$$\times (a^2 - \ell^2 + 3) \cos(2\theta_2 - \theta - 4\theta_1) - \frac{12 a^6 v_1 r}{\ell^6 r_1^5} \cos(2\theta_2 - \theta - 5\theta_1) \Big]$$

$$+ \frac{2 u_m \delta_3}{\alpha_m + 1} \Big[\sin 2\theta_2 + \frac{6 a^6 v_1}{\ell^6 r_1^4} \sin 4\theta_1 + \frac{4 v_1 a^4}{\ell^5 r_1^3} (a^2 - \ell^2 + 3) \sin 3\theta_1$$

$$+ \frac{2a^2 v_1}{\ell^4 r_1^2} (3a^2 - 2\ell^2 + 3) \sin 2\theta_1 - \frac{2a^4 v_1}{\ell^3 r_1^3} \sin(2\theta_2 - 3\theta_1)$$

$$- \frac{3a^4 v_1}{\ell^4 r_1^4} (2a^2 - 2\ell^2 + 3) \sin(2\theta_2 - 4\theta_1)$$

$$- \frac{12 a^6 v_1}{\ell^5 r_1^5} \sin(2\theta_2 - 5\theta_1) - (v_2 - v_1) \frac{a^2}{\ell^2 r_1^2} \sin(2\theta_2 - 2\theta_1)$$

$$-(v_1 - v_2) \frac{a^2}{\ell^2 r^2} \sin(2\theta_2 - 2\theta) + \frac{2 v_1 a^2 r}{\ell^4 r_1^3} (3a^2 - 2\ell^2 + 3)$$

$$\times \sin(2\theta_2 - \theta - 3\theta_1) + \frac{6a^4 v_1 r}{\ell^4 r_1^4} (a^2 - \ell^2 + 3) \sin(2\theta_2 - \theta - 4\theta_1)$$

$$+ \frac{12 a^6 v_1 r}{\ell^6 r_1^5} \sin(2\theta_2 - \theta - 5\theta_1) \Big].$$

The jump in hoop stress across the boundary of the inclusion is

$$(\sigma_3)_m\Big|_{r_2=1} - (\sigma_3)_i\Big|_{r_2=1} = 2(\lambda_m+\mu_m)\frac{\alpha_m-1}{\alpha_m+1}(\delta_1+\delta_2) - \frac{4\mu_m}{\alpha_m+1}(\delta_1-\delta_2)\cos 2\theta_2 - \frac{8\mu_m}{\alpha_m+1}\delta_3 \sin 2\theta_2 .$$

The ratio of hoop stresses at the point B (Fig. 10 p. 138) of the inclusion boundary calculated from inside and from the matrix is given by

$$\frac{1-\dfrac{2a^2\nu_1(\ell^2-a^2)}{(\ell^2-\ell-a^2)^3}}{1+\dfrac{2a^2\nu_1(\ell^2-a^2)}{(\ell^2-\ell-a^2)^3}},$$

for the case $\delta_1=\delta_2$, $\delta_3=0$. The continuous normal and the tangential stresses across the boundary of the inhomogeneity are given by

$$(\sigma_r)_{r=a} = -(\lambda_m+\mu_m)(\delta_1+\delta_2)\left(\frac{\alpha_m-1}{\alpha_m+1}\right)\frac{1}{r_2^2}\cos(2\theta-2\theta_2)$$
$$-\frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1}\Big[\frac{2(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\frac{1}{\ell^2} - \frac{2(\nu_2-1)}{r_2^2}\cos 2\theta_2$$
$$-2(\nu_2-1)\frac{a}{r_2^3}\cos(\theta-3\theta_2) - 2(1-\nu_1)\frac{\ell}{r_2^3}\cos(2\theta-3\theta_2)$$
$$+(\nu_1-\nu_2)\frac{a^2}{\ell^2 r_2^2}\cos(2\theta-2\theta_2) - 2(\nu_1-\nu_2)\frac{a^2}{\ell r_2^3}\cos(2\theta-3\theta_2)$$

$$- 3(1-\nu_1)\frac{1}{r_2^4}\cos(2\theta-4\theta_2)\Big] + \frac{2\mu_m\delta_3}{\alpha_m+1}\Big[\frac{2(\nu_2-1)}{r_2^2}\sin 2\theta_2$$

$$- 2(\nu_2-1)\frac{a^2}{r_2^3}(\theta-3\theta_2) + (\nu_1-\nu_2)\frac{a^2}{\ell^2 r_2^2}\sin(2\theta-2\theta_2)$$

$$- 2(\nu_1-\nu_2)\frac{a^2}{\ell r_2^3}\sin(2\theta-3\theta_2) - 2(1-\nu_1)\frac{\ell}{r_2^3}$$

$$\times \sin(2\theta-3\theta_2) - 3(1-\nu_1)\frac{1}{r_2^4}\sin(2\theta-4\theta_2)\Big],$$

$$(\tau_{r\theta})_{r=a} = \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{1}{r_2^2}\sin(2\theta-2\theta_2)$$

$$- \frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1}\Big[(\nu_2-1)\frac{a}{r_2^3}\sin(\theta-3\theta_2)$$

$$- (\nu_1-\nu_2)\Big\{\frac{a^2}{\ell^2 r_2^2}\sin(2\theta-2\theta_2) - \frac{2a^2}{\ell r_2^3}\sin(2\theta-3\theta_2)\Big\}$$

$$+ (1-\nu_1)\Big\{\frac{2\ell}{r_2^3}\sin(2\theta-3\theta_2) + \frac{3}{r_2^4}\sin(2\theta-4\theta_2)\Big\}\Big]$$

$$- \frac{2\mu_m\delta_3}{\alpha_m+1}\Big[(\nu_2-1)\frac{a}{r_2^3}\cos(\theta-3\theta_2) - (\nu_1-\nu_2)$$

$$\times\Big\{\frac{a^2}{\ell^2 r_2^2}\cos(2\theta-2\theta_2) - \frac{2a^2}{\ell r_2^3}\cos(2\theta-3\theta_2)\Big\}$$

$$+ (1-\nu_1)\Big\{\frac{2\ell}{r_2^3}\cos(2\theta-3\theta_2) - \frac{3}{r_2^4}\cos(2\theta-4\theta_2)\Big\}\Big].$$

the hoop stress on the inhomogeneity boundary is

$$\left[(\sigma_\theta)_{r=a}\right]_{ih} = (\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{\alpha_m-1}{\alpha_m+1} \quad \frac{1-\nu_1}{r_2^2}\cos(2\theta-2\theta_2)$$

$$-\frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1}\left[\frac{2(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right.$$

$$-2(\nu_2-1)\left\{\frac{\cos 2\theta_2}{r_2^2}-\frac{a}{r_2^3}\cos(\theta-3\theta_2)\right\}$$

$$-(\nu_1-\nu_2)\left\{\frac{a^2}{\ell^2 r_2^2}\cos(2\theta-2\theta_2)-\frac{2a^2}{\ell r_2^3}\cos(2\theta-3\theta_2)\right\}$$

$$\left.+(1-\nu_1)\left\{\frac{2\ell}{r_2^3}\cos(2\theta-3\theta_2)+\frac{3}{r_2^4}\cos(2\theta-4\theta_2)\right\}\right]$$

$$+\frac{2\mu_m\delta_3}{\alpha_m+1}\left[\left\{\frac{2(\nu_2-1)}{r_2^2}\sin 2\theta_2+2(\nu_2-1)\frac{a}{r_2^3}\sin(\theta-3\theta_2)\right\}\right.$$

$$-(\nu_1-\nu_2)\left\{\frac{a^2}{\ell^2 r_2^2}\sin(2\theta-2\theta_2)-\frac{2a^2}{\ell r_2^3}\sin(2\theta-3\theta_2)\right\}$$

$$\left.+(1-\nu_1)\left\{\frac{2\ell}{r_2^3}\sin(2\theta-3\theta_2)+\frac{3}{r_2^4}\sin(2\theta-4\theta_2)\right\}\right],$$

and

$$\left[(\sigma_\theta)_{r=a}\right]_m = \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\left[\frac{4a^2\nu_1}{\ell^2 r_1^2}\cos 2\theta_1\right.$$

$$\left.+\frac{1-\nu_1}{r_2^2}\cos(2\theta-2\theta_2)\right]-\frac{\mu_m(\delta_1-\delta_2)}{\alpha_m+1}$$

$$\times\left[\frac{4}{r_2^2}\cos 2\theta_2+2(\nu_2-1)\left\{2\frac{\cos 2\theta_2}{r_2^2}\right.\right.$$

$$\left.+\frac{a}{r_2^3}\cos(\theta-3\theta_2)\right\}-(\nu_1-\nu_2)\left\{\frac{a^2}{\ell^2 r_2^2}\cos(2\theta-3\theta_2)\right.$$

$$- \frac{2a^2}{\ell r_2^3} \cos(2\theta - 3\theta_2)\Big\} + (1-\nu_1)\Big\{\frac{2\ell}{r_2^3}\cos(2\theta-3\theta_2) + \frac{3}{r_2^4}\cos(2\theta-4\theta_2)\Big\}$$

$$- \frac{2(\nu_2-1)}{\ell^2(2\beta+\alpha_{ih}-1)} + 4(3a^2-2\ell^2+3)\frac{\nu_1 a^2}{\ell^4 r_1^2}\cos 2\theta_1 + \frac{8a^4\nu_1}{\ell^5 r_1^3}(a^2-\ell^2+3)\cos 3\theta_1$$

$$+ \frac{12a^6\nu_1}{\ell^6 r_1^4}\cos 4\theta_1\Big] - \frac{2\mu_m\delta_3}{\alpha_m+1}\Big[\frac{4\sin 2\theta_2}{r_2^2} + 2(\nu_2-1)\Big\{\frac{\sin 2\theta_2}{r_2^2}$$

$$- \frac{2a}{r_2^3}\sin(\theta-3\theta_2)\Big\} + (\nu_1-\nu_2)\Big\{\frac{a^2}{\ell^2 r_2^2}\sin(2\theta-2\theta_2) - \frac{2a^2}{\ell r_2^3}\sin(2\theta-3\theta_2)\Big\}$$

$$-(1-\nu_1)\Big\{\frac{2\ell}{r_2^3}\sin(2\theta-3\theta_2) + \frac{3}{r_2^3}\sin(2\theta-4\theta_2)\Big\} - 4(3a^2-2\ell^2+3)$$

$$\times \frac{a^4\nu_1}{\ell^4 r_1^2}\sin 2\theta_1 - 8(a^2-\ell^2+3)\frac{a^4\nu_1}{\ell^5 r_1^3}\sin 3\theta_1 - \frac{12a^6\nu_1}{\ell^6 r_1^4}\sin 4\theta_1\Big].$$

The ratio of hoop stresses at the point A (Fig. 10 p. 138) of the boundary of inhomogeneity calculated from inside and outside is given by $(\delta_1 = \delta_2,\ \delta_3 = 0)$

$$\frac{1-\nu_1}{(\ell-a)^2} \Big/ \left(\frac{4\nu_1}{\ell^2} + \frac{(1-\nu_1)}{(\ell-a)^2}\right) .$$

Integrating the equations (131) - (136) and fixing the constants of integration in such a way that the displacements remain continuous across the boundaries, both of the inclusion and the inhomogeneity, we obtain

$$\phi_{ih}(z) = \frac{-\mu_m(\delta_1-\delta_2+2i\delta_3)(\nu_2-1)}{\alpha_m+1}\Big\{\frac{1}{z_2^2} + \frac{(\beta-1)z}{2\ell^2(2\beta+\alpha_{ih}-1)}$$

$$-\frac{(\beta-1)z}{2\ell^2(\alpha_{ih}+1)}\Big\} - \frac{\mu_m(\delta_1-\delta_2-2i\delta_3)}{\alpha_m+1}\,\frac{(\nu_2-1)(\beta-1)z}{2\ell^2}$$

$$\times\Big\{\frac{1}{2\beta+\alpha_{ih}-1} + \frac{1}{\alpha_{ih}+1}\Big\}, \qquad (121)$$

$$
\begin{aligned}
\psi_{ih}(z) = {} & \frac{\alpha_m - 1}{\alpha_m + 1} (\delta_1 + \delta_2)(\lambda_m + \mu_m) \left(\frac{\nu_2 - 1}{z_2} - \frac{\beta \alpha_m \nu_1}{\ell} \right) \\
& + \frac{\mu_m (\delta_1 - \delta_2 + 2i\delta_3)}{\alpha_m + 1} \Bigg[(\nu_2 - \nu_1) \left(\frac{a^2}{\ell^2 z_2} - \frac{a^2}{\ell z_2^2} \right) \\
& + (1 - \nu_1) \left(\frac{\ell}{z_2^2} + \frac{1}{z_2^3} \right) + \frac{\beta \alpha_m \nu_1}{\ell^3} (2a^2 - \ell^2 + 1) \Bigg] \\
& - \frac{\mu_m (\delta_1 - \delta_2 - 2i\delta_3)}{\alpha_m + 1} \, \frac{\beta \nu_2}{\ell} ,
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
\phi_m(z) = {} & - \nu_1 \frac{\alpha_m - 1}{\alpha_m + 1} (\lambda_m + \mu_m)(\delta_1 + \delta_2) \frac{a^2}{\ell^2 z_1} \\
& + \frac{\mu_m (\delta_1 - \delta_2 - 2i\delta_3)}{\alpha_m + 1} \Bigg[\frac{a^2 \nu_1}{\ell^4 z_1} (3a^2 - 2\ell^2 + 3) + \frac{a^6 \nu_1}{\ell^6 z_1^3} \\
& + \frac{a^4 \nu_1}{\ell^5 z_1^2} (a^2 - \ell^2 + 3) \Bigg] + \frac{\mu_m (\delta_1 - \delta_2 + 2i\delta_3)}{\alpha_m + 1} \, \frac{1}{z_2} ,
\end{aligned}
\tag{123}
$$

$$
\begin{aligned}
\psi_m(z) = {} & - \frac{\alpha_m - 1}{\alpha_m + 1} (\lambda_m + \mu_m)(\delta_1 + \delta_2) \left(\frac{\nu_1 a^2}{\ell z_1^2} - \frac{\nu_1}{z_1} + \frac{\nu_1}{z} + \frac{1}{z_2} \right) \\
& + \frac{\mu_m (\delta_1 - \delta_2 - 2i\delta_3)}{\alpha_m + 1} \Bigg[(\nu_2 - \nu_1) \frac{a^2}{\ell^2 z_1} + \frac{a^4 \nu_1}{\ell^3 z_1^2} \\
& + (2a^2 - 2\ell^2 + 3) \frac{a^4 \nu_1}{\ell^4 z_1^3} + \frac{3 a^6 \nu_1}{\ell^5 z_1^4} + \left\{ \nu_1 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1} \right\} \frac{a^2}{\ell^2 z} \Bigg] \\
& + \frac{\mu_m (\delta_1 - \delta_2 + 2i\delta_3)}{\alpha_m + 1} \left[\left\{ \nu_2 - \frac{(\nu_2 - 1)(\beta - 1)}{2\beta + \alpha_{ih} - 1} \right\} \frac{a^2}{\ell^2 z} + \frac{\ell}{z_2^2} + \frac{1}{z_2^3} \right] ,
\end{aligned}
\tag{124}
$$

$$\Phi_i(z) = \frac{(\lambda_m+\mu_m)(\delta_1+\delta_2)}{\alpha_m+1} z_2 - (\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{\alpha_m-1}{\alpha_m+1}\frac{a^2\nu_1}{\ell^2 z_1}$$
$$+ \nu_1\frac{\mu_m(\delta_1-\delta_2-2i\delta_3)}{\alpha_m+1}\left[\frac{a^6}{\ell^6 z_1^3} + (a^2-\ell^2+3)\frac{a^4}{\ell^5 z_1^2}\right.$$
$$\left. + (3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1}\right],$$

(125)

$$\Psi_i(z) = -(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{\alpha_m-1}{\alpha_m+1}\nu_1\left(\frac{a^2}{\ell z_1^2} - \frac{1}{z_1} + \frac{1}{z}\right)$$
$$- \frac{\ell(\lambda_m+\mu_m)(\delta_1+\delta_2)}{\alpha_m+1} + \frac{\mu_m(\delta_1-\delta_2-2i\delta_3)}{\alpha_m+1}$$
$$\times\left[-\alpha_m z_2 + (\nu_2-\nu_1)\frac{a^2}{\ell^2 z_1} + \left\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ch}-1}\right\}\frac{a^2}{\ell^2 z}\right.$$
$$\left. + \frac{a^4\nu_1}{\ell^3 z_1^2} + (2a^2-2\ell^2+3)\frac{a^4\nu_1}{\ell^4 z_1^2} + \frac{3a^6\nu_1}{\ell^5 z_1^4}\right]$$
$$+ \frac{\mu_m(\delta_1-\delta_2+2i\delta_3)}{\alpha_m+1}\left[\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ch}-1}\right]\frac{a^2}{\ell^2 z}.$$

(126)

The displacements at points of the three regions are obtained from their respective potential functions with the help of (2,). Numerical calculations were carried out for stresses at the boundaries of the inclusion in the case of generalised plane stress taking (Poisson's ratio 1/3), $\delta_3 = 0$. Table 2 gives values of various stresses at points of the boundary of inhomogeneity for different values of the parameters β, distance ℓ and then inclusion radius a

At the point A the variation of all the stress components has been plotted in Figs. 34-36 p. 151-152.

Table 3 contains values of stress components at points of the inclusion boundary for various cases. Figs. 31-33 p. 150-151 show the behaviour of different stress components at the point B with changes in the parameters β, $\frac{a}{\lambda}$ and ℓ .

CHAPTER XII

TWO DEFORMING INHOMOGENEITIES

In the last chapter of this thesis we propose to give the solution of the problem of two deforming inhomogeneities. This involves two regions in the medium which have a tendency to undergo dimensional changes simultaneously. So far there was only one such region. The elastic constants of these regions could be different from the remaining surrounding material. The solution is obtained by making use of the results of two of the earlier chapters by an interesting use of the technique of superposition.

Choosing the reference frame as before the two inhomogeneities have the equations $z\bar{z} \leq a^2$ and $(z-\ell)(\bar{z}-\ell) \leq 1$ (Fig. 37 p.153) and the remaining portion of the complex plane represents the matrix.

For convenience, the three regions will be called regions 1, 2, and 3 respectively (Fig. 37 p. 153). Let the dimensional changes which the two inhomogeneities might have undergone in the absence of the matrix be given in terms of the cartesian components by the following equations

$$\left.\begin{aligned} u &= \delta_1 x + \delta_3 y , \\ v &= \delta_3 x + \delta_2 y , \end{aligned}\right\} \tag{127}$$

for inhomogeneity 1;

$$\left.\begin{aligned} u &= \delta_4 (x - \ell) + \delta_6 y , \\ v &= \delta_6 (x - \ell) + \delta_5 y , \end{aligned}\right\} \tag{128}$$

for inhomogeneity 2,

In this chapter it appears to be necessary to reiterate that the words inclusion and inhomogeneity are to be used in a definitive sense. Inclusion is a material region which has the same elastic properties as the matrix, while the inhomogeneity is one which may have different elastic properties. The solutions of the following problems obtained in earlier chapters form the starting point of the solution of the present problem.

(i) The problem of an oversize circular inclusion in an infinite medium containing an inhomogeneity which does not tend to undergo a deformation (chapter XI).

(ii) In an elastic infinite medium an inhomogeneity tends to undergo a deformation (Chapter VII).

In Fig. 38 p. 154 , a flow chart is given indicating the process which leads to the solution. This helps in understanding and visualising the procedure.

In the first step we solve the following problem : Region 2 is occupied by an inclusion which tends to undergo a deformation characterised by (128) and the region 1 contains an inhomogeneity which tends to undergo a deformation characterised by (127). It is required to determine the complex functions which would give the stresses in the system.

The solution is obtained by direct superposition of the results of (i) and (ii) mentioned above and given in equations (115) - (120) and (72) - (75) , replacing of course δ_1 , δ_2, δ_3 in (115) - (120) by δ_4 , δ_5 and δ_6 in that order. The complex functions obtained are given below. The subscripts indicate the region which they pertain to. ($z_1 = z - a^2/\ell$, $z_2 = z - \ell$, $z_3 = z - \ell + a^2/\ell$)

$$
\begin{aligned}
{}^1\phi_1'(z) = {} & \frac{1-\alpha_{ih}}{2(2\beta+\alpha_{ih}-1)} (\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) + \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6) \\
& \times (\nu_2-1)\left(\frac{1}{z_2^2} - \frac{\beta-1}{2\beta+\alpha_{ih}-1}\right) - \frac{\mu_m}{\alpha_m+1} \\
& (\delta_4-\delta_5-2i\delta_6)\ \frac{(\nu_2-1)(\beta-1)}{2\ell^2(2\beta+\alpha_{ih}-1)} ,
\end{aligned} \qquad (129)
$$

$$
\begin{aligned}
{}^1\psi_1'(z) = {} & \frac{\alpha_m\nu_1+1}{\alpha_m+1}\ \mu_{ih}(\delta_1-\delta_2-2i\delta_3) + \frac{\alpha_m-1}{\alpha_m+1}(1-\nu_1)(\lambda_m+\mu_m) \\
& \times (\delta_4+\delta_5)\frac{1}{z_2^2} + (\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6) \\
& \times \left(\frac{a^2}{\ell^2 z_2^2} - \frac{2a^2}{\ell z_2^3}\right) - (1-\nu_1)\,\mu_m(\delta_4-\delta_5+2i\delta_6) \\
& \times \left(\frac{2\ell}{z_2^3} + \frac{3}{z_2^4}\right) ,
\end{aligned} \qquad (130)
$$

$$
\begin{aligned}
{}^{1}\Phi_2'(z) = &- \frac{1+\nu_1\alpha_m}{\alpha_m+1}\,\frac{a^2}{z^2}\,\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\\
&- \frac{\alpha_m-1}{2(\alpha_m+1)}(\lambda_m+\mu_m)(\delta_4+\delta_5) - \frac{a^2\nu_1}{\ell^2 z_1^2}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{\alpha_m-1}{\alpha_m+1}\\
&- \nu_1\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\Big[(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2}\\
&+ 2(a^2-\ell^2+3)\frac{a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4}\Big],
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
{}^{1}\Psi_2'(z) = &\frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}\,\frac{a^2}{z^2}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\\
&\times \frac{3a^4\mu_{ih}}{z^4}(\delta_1-\delta_2+2i\delta_3) + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\\
&\times \nu_1\left(\frac{2a^2}{\ell z_1^3}+\frac{1}{z^2}-\frac{1}{z_1^2}\right) - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\\
&\times\Big[-1+(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_1^2} + \Big\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\Big\}\frac{a^2}{\ell^2 z^2}\\
&+ \frac{2a^4\nu_1}{\ell^3 z_1^3} + 3(2a^2-2\ell^2+3)\frac{a^4\nu_1}{\ell^4 z_1^4} + 12\frac{a^6\nu_1}{\ell^5 z_1^5}\Big]\\
&- \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\Big\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\Big\}\frac{a^2}{\ell^2 z^2},
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
{}^{1}\Phi_3'(z) = & -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} + \nu_1\frac{a^2}{\ell^2 z_1^2}\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5) \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2} - \frac{\nu_1\alpha_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6) \\
& \times\left[(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2} + 2(a^2-\ell^2+3)\frac{a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^4 z_1^4}\right]
\end{aligned}
\tag{133}
$$

$$
\begin{aligned}
{}^{1}\Psi_3'(z) = & \frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\frac{a^2}{z^2} - \frac{(1+\nu_1\alpha_m)}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4} \\
& + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\left(\frac{1}{z_2^2} + \frac{2a^2\nu_1}{\ell z_1^3} - \frac{\nu_1}{z_1^2} + \frac{\nu_1}{z^2}\right) \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_1^2} + \left\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z^2}\right. \\
& \left. + \frac{2a^2\nu_1}{\ell^3 z_1^3} + (2a^2-2\ell^2+3)\frac{3a^4\nu_1}{\ell^4 z_1^4} + \frac{12a^6\nu_1}{\ell^5 z_1^5}\right] \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left[\left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z^2} + \frac{2\ell}{z_2^3} + \frac{3}{z_2^4}\right].
\end{aligned}
\tag{134}
$$

It should be noted that these functions directly give the stress fields in their respective regions. We have added the initially existing fields if any.

In the second step we solve the following problem: Region 2 is occupied by an inhomogeneity which tends to undergo a deformation given by (128). The region 1 is occupied by an inclusion which tends to undergo a deformation given by (127). Complex functions for the elastic fields developed are to be determined.

The solution is obtained from the functions (129) - (134) given above. This can be done by a suitable transformation of the coordinate system- shift the origin to $(\ell, 0)$ which is the centre of the region 2, and then rotate through an angle 180°. This is to be followed by an interchange of the roles of $\delta_1, \delta_2, \delta_3$ with δ_4, δ_5 and δ_6 and making the radii of inhomogeneity and the inclusion 1 and 'a' respectively. The formulae (3) - (6) are used to obtain the complex functions in the new coordinate system from the old ones. The results are given below: $\left(z_1 = z - a^2/\ell,\ z_2 = z - \ell,\ z_3 = z - (\ell - a^2/\ell)\right)$:

$$
\begin{aligned}
{}^2\phi_1'(z) = &- \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2} - \frac{\alpha_m-1}{2(\alpha_m+1)}(\lambda_m+\mu_m)(\delta_1+\delta_2) \\
&+ \nu_1\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_3^2} - \frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
&\times\left[(3-2\ell^2+3a^2)\frac{a^4}{\ell^4 z_3^2} + 2(1-\ell^2+3a^2)\frac{a^2}{\ell^5 z_3^2} + \frac{3a^4}{\ell^6 z_3^4}\right],
\end{aligned}
\tag{135}
$$

$$
\begin{aligned}
{}^2\psi_1'(z) = &\frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}\frac{1}{z_2^2}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih} \\
&\times(\delta_4-\delta_5+2i\delta_6)\left(\frac{3}{z_2^4}+\frac{2\ell}{z_2^3}\right) + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2) \\
&\times\nu_1\left(\frac{a^2}{z_2^2}-\frac{a^2}{z_3^2}\right) - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[-1+(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_3^2}\right. \\
&+\left\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2} + (2-2\ell^2+3a^2)\frac{2a^2\nu_1}{\ell^3 z_3^3} \\
&\left.+(4-4\ell^2+9a^2)\frac{3a^2\nu_1}{\ell^4 z_3^4}\right] - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3) \\
&\times\left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2},
\end{aligned}
\tag{136}
$$

$$
{}^{2}\phi_2'(z) = \frac{1-\alpha_{ch}}{2\beta+\alpha_{ch}-1}(\lambda_{ch}+\mu_{ch})(\delta_4+\delta_5) + \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)
$$

$$
\times(\nu_2-1)\left\{\frac{a^2}{z^2} - \frac{(\beta-1)\,a^2}{2\ell^2(2\beta+\alpha_{ch}-1)}\right\}
$$

$$
- \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\frac{(\nu_2-1)(\beta-1)\,a^2}{2\ell^2(2\beta+\alpha_{ch}-1)}, \qquad (137)
$$

$$
{}^{2}\psi_2'(z) = \frac{\alpha_m\nu_1+1}{\alpha_m+1}\mu_{ch}(\delta_4-\delta_5-2i\delta_6) + \frac{\alpha_m-1}{\alpha_m+1}(1-\nu_1)
$$

$$
\times(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2} + (\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)
$$

$$
\times\left(\frac{a^2}{\ell^2 z^2} + \frac{2a^2}{\ell z^3}\right) - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)
$$

$$
\times\left\{\frac{2\ell a^2}{z^3}(\nu_1-\nu_2) + \frac{3(1-\nu_1)\,a^4}{z^4}\right\}, \qquad (138)
$$

$$
{}^{2}\phi_3'(z) = -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ch}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}
$$

$$
+ \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_3^2}
$$

$$
- \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} - \frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)
$$

$$
\times\left[(3-2\ell^2+3a^2)\frac{a^2}{\ell^4 z_3^2} - 2(1-\ell^2+3a^2)\frac{a^2}{\ell^5 z_3^2} + \frac{3a^4}{\ell^6 z_3^4}\right], \qquad (139)
$$

$$
\begin{aligned}
{}^2\psi_3'(z) = {} & \frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}\,\frac{1}{z_2^2}\,(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2) - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\,\mu_{ih} \\
& \times(\delta_4-\delta_5+2i\delta_6)\left(\frac{3}{z_2^4}+\frac{2\ell}{z_3^3}\right) + (\lambda_m+\mu_m)(\delta_1+\delta_2)\,\frac{\alpha_m-1}{\alpha_m+1} \\
& \times\left(\frac{a^2}{z^2}+\frac{\nu_1 a^2}{z_2^2}-\frac{\nu_1 a^2}{z_3^2}\right) - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
& \times\Bigg[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_3^2} + \left\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2} \\
& + (2-2\ell^2+3a^2)\,\frac{2a^2\nu_1}{\ell^3 z_3^3} + 3\,(4-4\ell^2+9a^2)\,\frac{a^2\nu_1}{\ell^4 z_3^4}\Bigg] \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\Bigg[\left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2} \\
& + 3\,\frac{a^4}{z^4}\Bigg]. \qquad (140)
\end{aligned}
$$

In the next step we solve the following problem : Two inclusions occupy regions 1 and 2. They tend to undergo deformations given by (121) and (128) respectively.

The complex functions which describe the effect of these inclusions can be obtained from (129) - (134) by equating the constants of region 1 and 2 with those of 3. Thus we have

$$
{}^3\phi_1'(z) = \frac{\lambda_m+\mu_m}{2(\alpha_m+1)}(\delta_1+\delta_2)(1-\alpha_m) - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\,\frac{1}{z_2^2}\,, \qquad (141)
$$

$$
\begin{aligned}
{}^3\psi_1'(z) = {} & \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\,\frac{1}{z_2^2} \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left(\frac{2\ell}{z_2^3}+\frac{3}{z_2^4}\right), \qquad (142)
\end{aligned}
$$

$$
\begin{aligned}
{}^{3}\phi_2'(z) = &-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} \\
&+\frac{1-\alpha_m}{2(\alpha_m+1)}(\delta_4+\delta_5)(\lambda_m+\mu_m),
\end{aligned}
\tag{143}
$$

$$
\begin{aligned}
{}^{3}\psi_2'(z) = &\frac{\alpha_m-1}{\alpha_m+1}\frac{a^2}{z^2}(\lambda_m+\mu_m)(\delta_1+\delta_2) \\
&-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4} \\
&+\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6),
\end{aligned}
\tag{144}
$$

$$
\begin{aligned}
{}^{3}\Phi_3'(z) = &-\frac{\mu_m}{\alpha_m+1}\Big\{(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} \\
&+(\delta_4-\delta_5+2i\delta_6)\frac{a^2}{z_2^2}\Big\},
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
{}^{3}\psi_3'(z) = &\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2} \\
&-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4} \\
&+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{1}{z_2^2} \\
&-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{2\ell}{z_2^3}+\frac{3}{z_2^4}\right).
\end{aligned}
\tag{146}
$$

We are now in a position to obtain the solution of the problem of this chapter i.e. the problem of two deforming inhomogeneities. Following the flow chart on page 154 we obtain the desired functions in the following manner.

$$
\begin{aligned}
{}^4\phi_1'(z) = {} & {}^1\phi_1'(z) + {}^2\phi_1'(z) - {}^3\phi_1'(z) \\
= {} & \frac{1-\alpha_{ch}}{2(2\beta+\alpha_{ch}-1)}(\mu_{ch}+\lambda_{ch})(\delta_1+\delta_2) + \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6) \\
& \times\left\{\frac{\nu_2}{z_2^2} - \frac{(\nu_2-1)(\beta-1)}{(2\beta+\alpha_{ch}-1)2\ell^2}\right\} - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ch}(\delta_4-\delta_5+2i\delta_6)\frac{a^2}{z_2^2} \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\frac{(\nu_2-1)(\beta-1)}{2\ell^2(2\beta+\alpha_{ch}-1)} + \frac{\nu_1(\alpha_m-1)}{\alpha_m+1} \\
& \times(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_3^2} - \frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
& \times\left[(3-2\ell^2+3a^2)\frac{a^2}{\ell^4 z_3^2} - (1-\ell^2+3a^2)\frac{2a^2}{\ell^5 z_3^3} + \frac{3a^4}{\ell^6 z_3^4}\right],
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
{}^4\psi_1'(z) = {} & {}^1\psi_1'(z) + {}^2\psi_1'(z) - {}^3\psi_1'(z) \\
= {} & \frac{\alpha_m\nu_1+1}{\alpha_m+1}\mu_{ch}(\delta_1-\delta_2-2i\delta_3) - \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m) \\
& \times(\delta_4+\delta_5)\frac{\nu_1}{z_2^2} + (\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6) \\
& \times\left(\frac{a^2}{\ell^2 z_2^2} - \frac{2a^2}{\ell z_2^3}\right) + \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\nu_1\left(\frac{2\ell}{z_2^3} + \frac{3}{z_2^4}\right) \\
& + \frac{\alpha_{ch}-1}{2\beta+\alpha_{ch}-1}(\delta_4+\delta_5)\frac{1}{z_2^2} - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ch}(\delta_4-\delta_5-2i\delta_6) \\
& \times\left(\frac{3}{z_2^4} + \frac{2\ell}{z_2^3}\right) + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\nu_1\left(\frac{a^2}{z_2^2} - \frac{a^2}{z_3^2}\right) \\
& - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_3^2}\right.
\end{aligned}
$$

$$+\left\{\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2}+2(z-2\ell^2+3a^2)\frac{a^2\nu_1}{\ell^3 z_3^3}-\frac{9a^4\nu_1}{\ell^4 z_3^4}\Bigg]$$

$$-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left\{\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2}\ , \tag{148}$$

$$\begin{aligned}
{}^4\phi_2'(z) = {} & {}^1\phi_2'(z)+{}^2\phi_2'(z)-{}^3\phi_2'(z)\\
= {} & \frac{1-\alpha_{ih}}{2(2\beta+\alpha_{ih}-1)}(\lambda_{ih}+\mu_{ih})(\delta_4+\delta_5)+\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\\
& \times\left\{\frac{\nu_2 a^2}{z^2}-\frac{a^2(\nu_2-1)(\beta-1)}{2\ell^2(2\beta+\alpha_{ih}-1)}\right\}-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\\
& \times\frac{a^2(\nu_2-1)(\beta-1)}{2\ell^2(2\beta+\alpha_{ih}-1)}-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2}\\
& +\frac{\alpha_m-1}{\alpha_m+1}\nu_1(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{a^2}{\ell^2 z_1^2}\\
& -\frac{\nu_1\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\Bigg[(3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2}\\
& +2(a^2-\ell^2+3)\frac{a^4}{\ell^5 z_1^3}+\frac{3a^6}{\ell^6 z_1^4}\Bigg],
\end{aligned} \tag{149}$$

$$\begin{aligned}
{}^4\psi_2'(z) = {} & {}^1\psi_2'(z)+{}^2\psi_2'(z)-{}^3\psi_2'(z)\\
= {} & \frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}(\lambda_{ih}+\mu_{ih})(\delta_1+\delta_2)\frac{a^2}{z^2}-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\\
& \times\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4}+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\\
& \times\nu_1\left(\frac{2a^2}{\ell z_1^3}+\frac{1}{z^2}-\frac{1}{z_1^2}\right)-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\times
\end{aligned}$$

$$\times\left[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_1^2} + \left\{ \nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1} \right\} \frac{a^2}{\ell^2 z^2} + \frac{2a^4\nu_1}{\ell^3 z_1^3} \right.$$

$$\left. + 3(2a^2-2\ell^2+3)\frac{a^4\nu_1}{\ell^4 z_1^4} + 12\frac{a^6\nu_1}{\ell^5 z_1^5} \right] - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)$$

$$\times\left\{ \nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1} \right\} \frac{a^2}{\ell^2 z^2} + \frac{\alpha_m\nu_1+1}{\alpha_m+1}\mu_{ih}(\delta_4-\delta_5-2i\delta_6)$$

$$- \frac{\alpha_m-1}{\alpha_m+1}\nu_1(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2}$$

$$+ (\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{a^2}{\ell^2 z^2} + \frac{2a^2}{\ell z^3}\right)$$

$$- \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left\{ 2\ell(\nu_1-\nu_2)\frac{a^2}{z^3} - \frac{3\nu_1 a^4}{z^4} \right\}, \qquad (150)$$

$$\overset{4}{\Phi}{}_3'(z) = \overset{1}{\Phi}{}_3'(z) + \overset{2}{\Phi}{}_3'(z) - \overset{3}{\Phi}{}_3'(z)$$

$$= -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} + \frac{\alpha_m-1}{\alpha_m+1}\nu_1(\lambda_m+\mu_m)$$

$$\times(\delta_4+\delta_5)\frac{a^2}{\ell^2 z_1^2} - \frac{\nu_1\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left\{ (3a^2-2\ell^2+3)\frac{a^2}{\ell^4 z_1^2} \right.$$

$$\left. + 2(a^2-\ell^2+3)\frac{a^4}{\ell^5 z_1^3} + \frac{3a^6}{\ell^6 z_1^4} \right\}$$

$$- \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}$$

$$+ \nu_1\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{\ell^2 z_3^2}$$

$$- \frac{\mu_m\nu_1}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(3-2\ell^2+3a^2)\frac{a^2}{\ell^4 z_3^2} \right.$$

$$\left. - 2(1-\ell^2+3a^2)\frac{a^2}{\ell^5 z_3^3} + \frac{3a^4}{\ell^6 z_3^4} \right], \qquad (151)$$

$$
\begin{aligned}
{}^{4}\psi_3'(z) &= {}^{1}\psi_3'(z) + {}^{2}\psi_3'(z) - {}^{3}\psi_3'(z) \\
&= \frac{\alpha_{ih}-1}{2\beta+\alpha_{ih}-1}(\lambda_{ih}+\mu_{ih})\left\{(\delta_1+\delta_2)\frac{a^2}{z^2} + (\delta_4+\delta_5)\frac{1}{z_2^2}\right\} \\
&\quad - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4} + (\lambda_m+\mu_m)(\delta_4+\delta_5) \\
&\quad \times \frac{\alpha_m-1}{\alpha_m+1}\left(\frac{2a^2\nu_1}{\ell z_1^3} + \frac{\nu_1}{z^2} - \frac{\nu_1}{z_1^2}\right) - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6) \\
&\quad \times \left[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_1^2} + \left\{\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z^2} + \frac{2a^2\nu_1}{\ell^3 z_1^3}\right. \\
&\quad \left. + 3(2a^2-2\ell^2+3)\frac{a^4\nu_1}{\ell^4 z_1^4} + \frac{12a^6\nu_1}{\ell^5 z_1^5}\right] \\
&\quad - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z^2} \\
&\quad - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_{ih}(\delta_4-\delta_5+2i\delta_6)\left(\frac{3}{z_2^4} + \frac{2\ell}{z_2^3}\right) \\
&\quad + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\left(\frac{a^2\nu_1}{z_2^2} - \frac{a^2\nu_1}{z_3^2}\right) \\
&\quad - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(\nu_2-\nu_1)\frac{a^2}{\ell^2 z_3^2}\right. \\
&\quad + \left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2} \\
&\quad \left. + (2-2\ell^2+3a^2)\frac{2a^2\nu_1}{\ell^3 z_3^3} + (4-4\ell^2+9a^2)\frac{3a^2\nu_1}{\ell^4 z_3^4}\right] \\
&\quad - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left\{\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_{ih}-1}\right\}\frac{a^2}{\ell^2 z_2^2}\,.
\end{aligned}
\tag{152}
$$

It may be worthwhile to point out that the solution to the problem of single inhomogeneity alone fails to yield the solution to the problem of two inhomogeneities through the process of superposition. The reason is as follows. Let us suppose that a deforming inhomogeneity is present at region 1, and region 2 and 3 are of the same material. The solution of such a problem is known and is given in chapter VIII. Next suppose a deforming inhomogeneity is now present at region 2, and region 1 and 3 are of the same material. The solution is known from the solution mentioned in the first case by merely shifting the origin. Therefore, for superposition to work, one has to make the added assumption that the region 1, 2 and 3 are of the same material. In other words simple superposition can yield result for two inclusions only and in fact for any number of inclusions.

The stress distribution in the various regions can now be obtained in the routine fashion. The expressions are not included here. It may be pointed out here that a check on the analysis can be made at each step by verifying that the normal and tangential stresses across the boundaries $z\bar{z}=a^2$ and $(z-\ell)(\bar{z}-\ell)=1$ are continuous. The hoop stress is discontinuous as it should be. Some numerical work was done on the IBM 1620 computer. A few figures (Figs. 39- 46 p. 155-157-a) have been included showing the variation of normal shearing and hoop stress on the boundary of the region 1 for plane stress case with the following data.

$$\alpha_{in}=\alpha_m=2,\ (\nu=\tfrac{1}{3});\ \ \beta=\tfrac{1}{2},1,2;\ \ a=1,\ \ell=3;\ \delta_4=\delta_1,\ \delta_5=\delta_2,\ \delta_3=\delta_6=0.$$

Table 4 contains the numbers from which the figures were drawn.

For the displacement fields, one can find out the functions ${}^4\phi_1(z)$, ${}^4\psi_1(z)$, ${}^4\phi_2(z)$, ${}^4\psi_2(z)$, ${}^4\phi_3(z)$ and ${}^4\psi_3(z)$ by integrating the corresponding functions in equations (147) - (151) with respect to z .

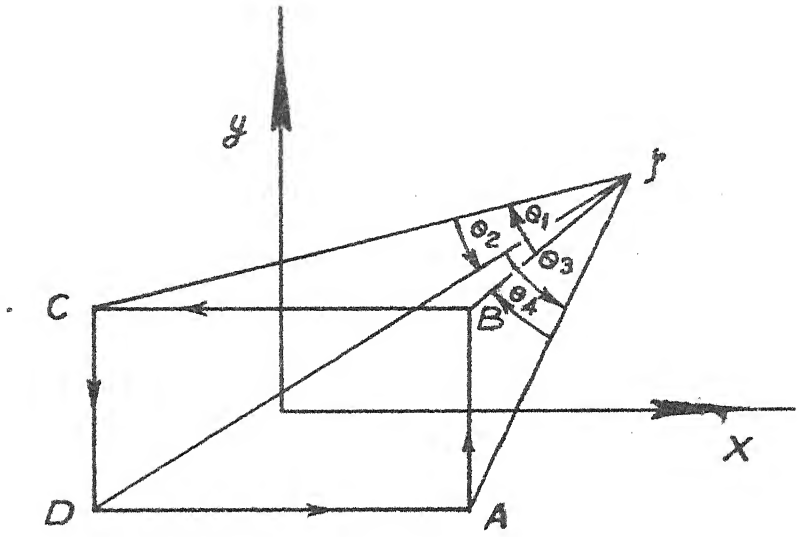


Figure 1. Rectangular inclusion and coordinate system.

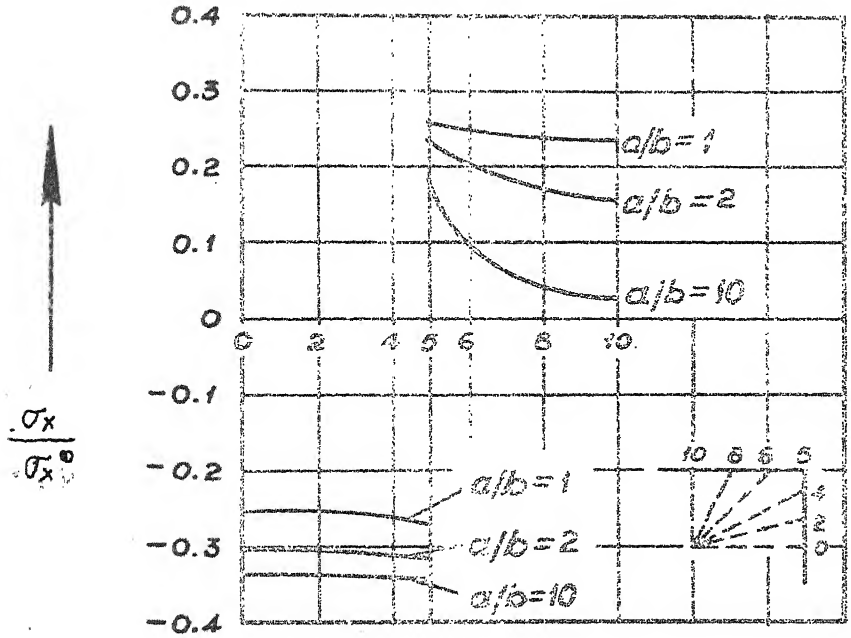


Figure 2. Variation of σ_x / σ_x^0 along the boundary.

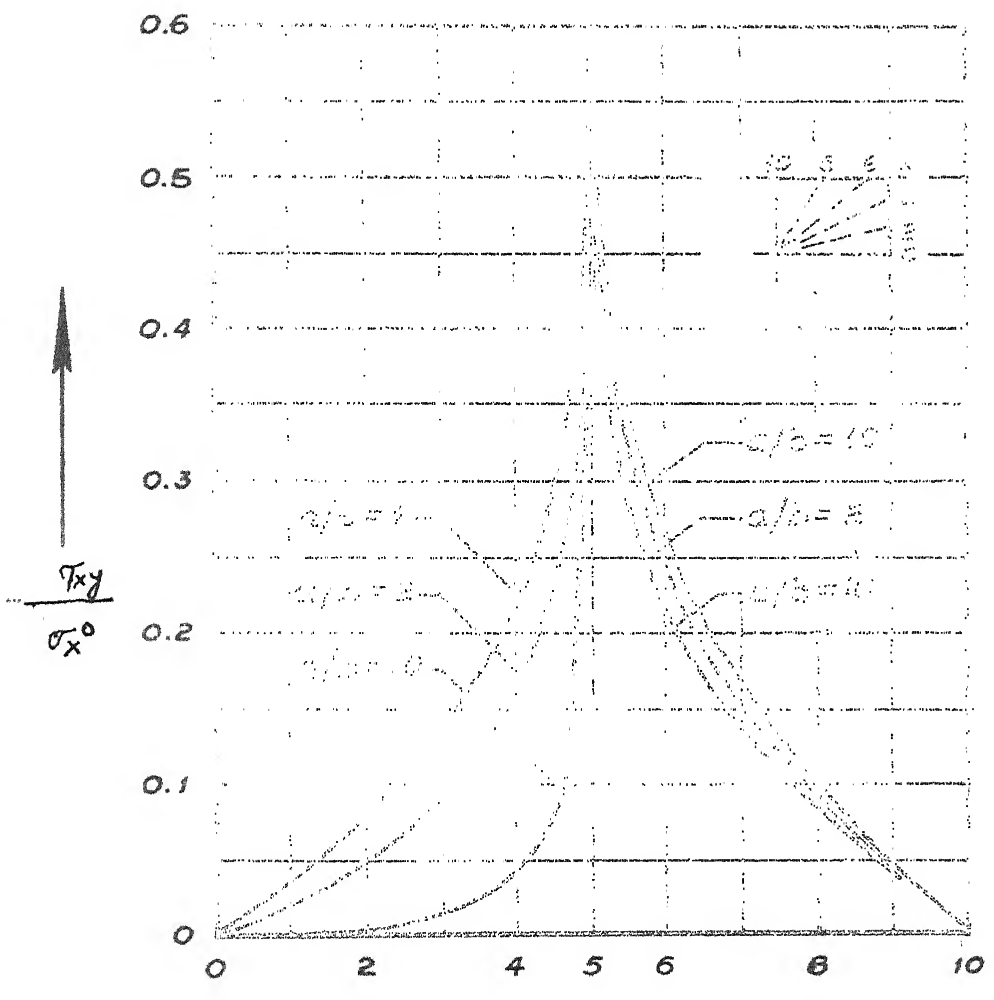


FIGURE 3. Shearing stress along the boundary.

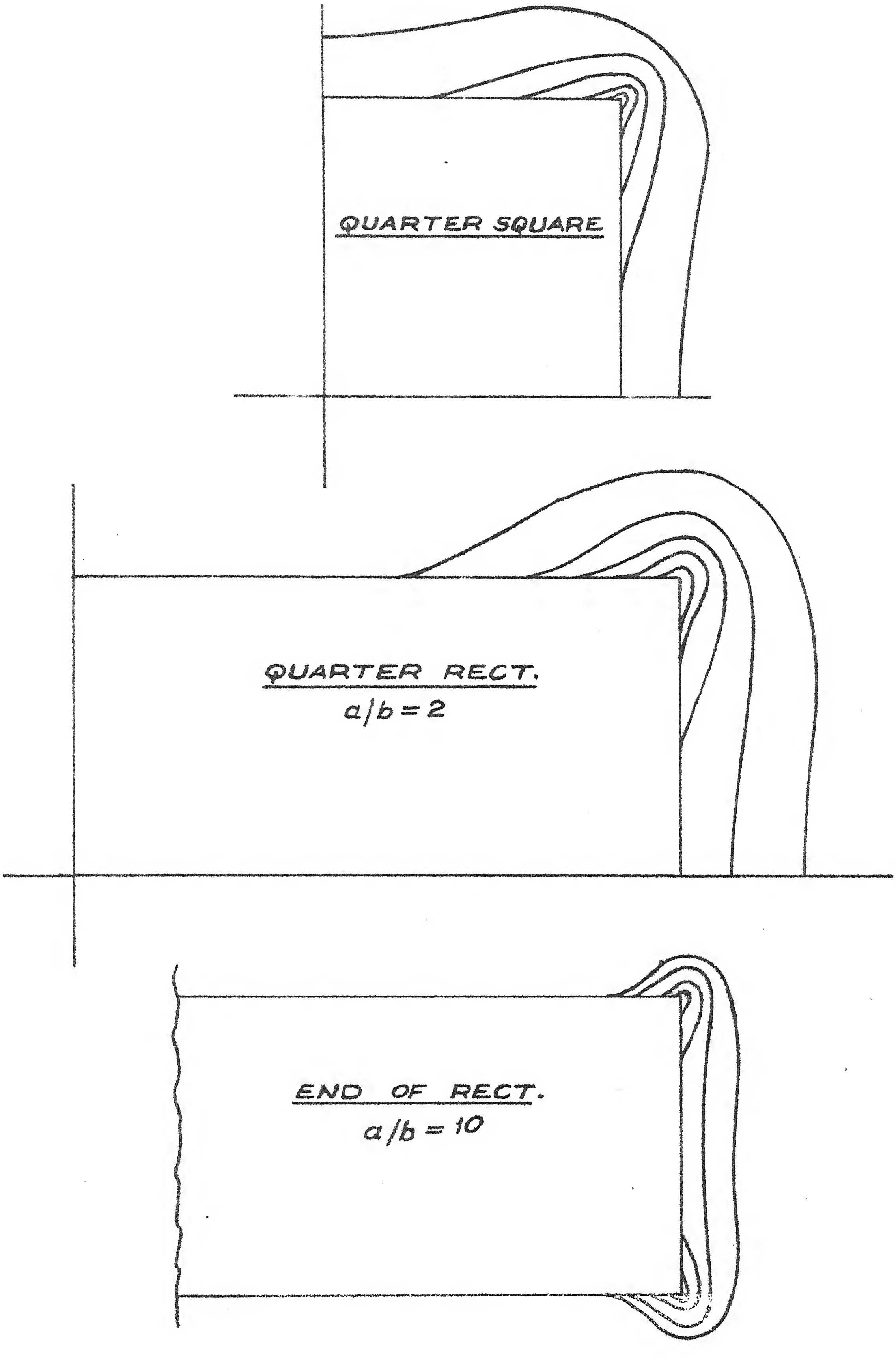


Figure 4. Lines of maximum shearing stress around the inclusion.

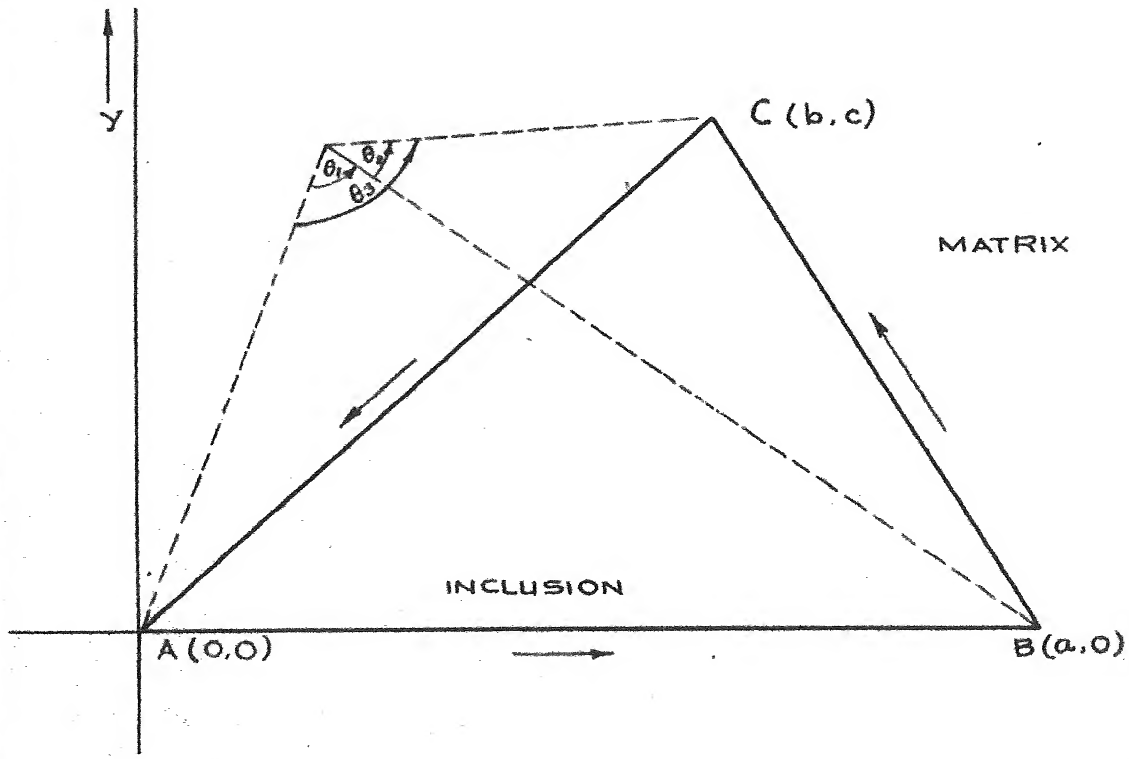


Figure 6. Triangular inclusion and coordinate system.

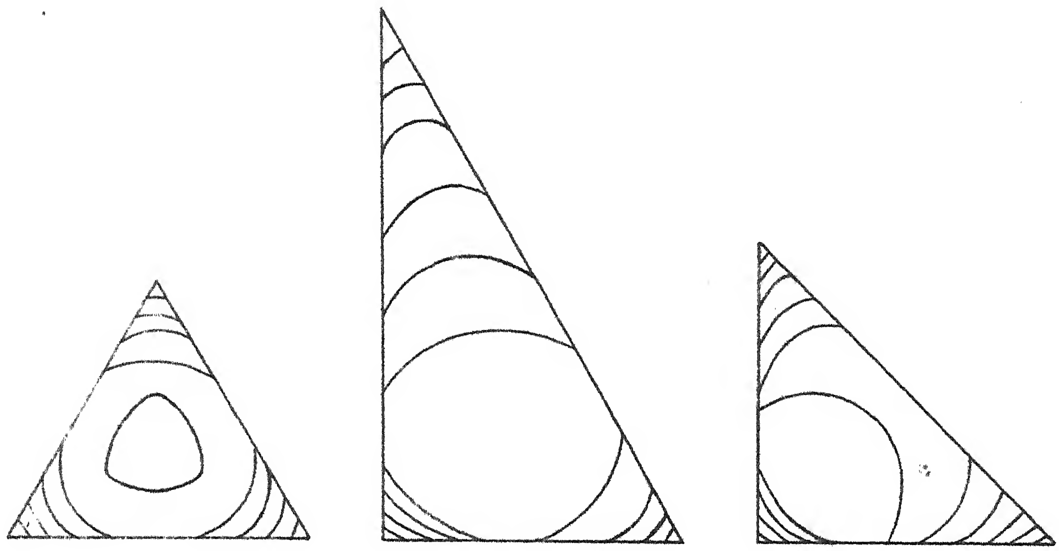


Figure 7. Lines of maximum shearing stress inside the triangular inclusion; $\delta_1 = \delta_2$, $\delta_3 = 0$.

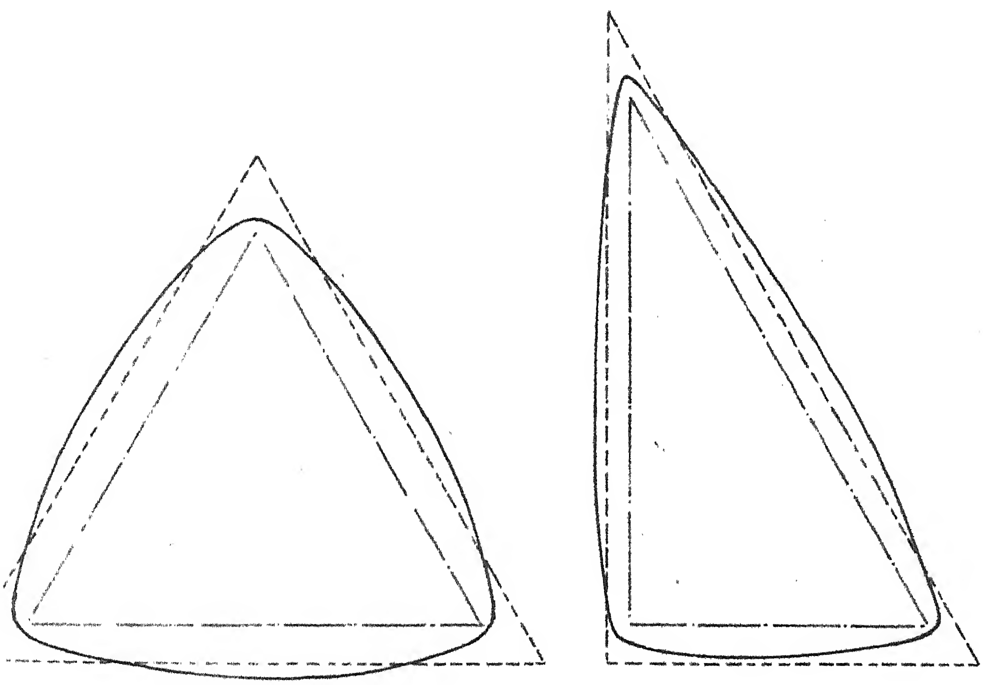


Figure 8. Schematic drawing of the equilibrium shape.

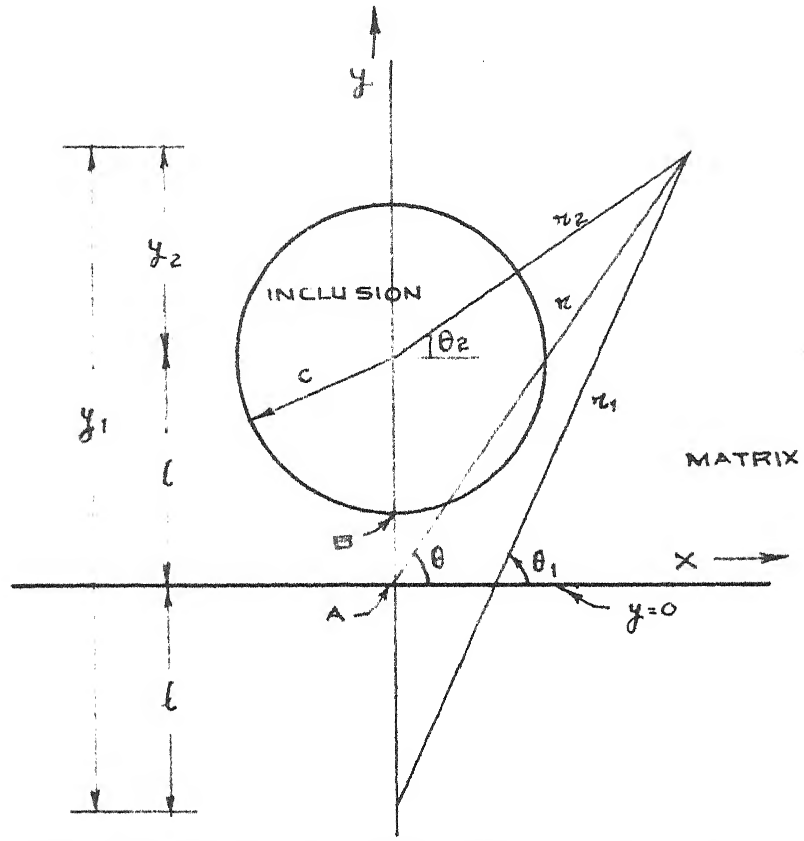


Figure 9. Circular inclusion in semi-infinite medium and coordinate system.

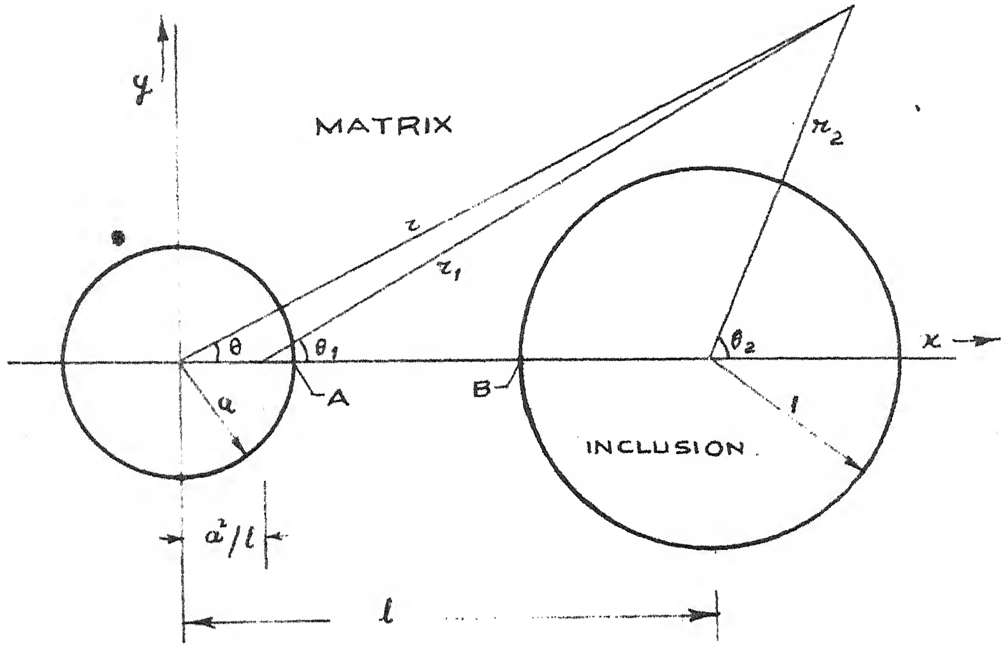
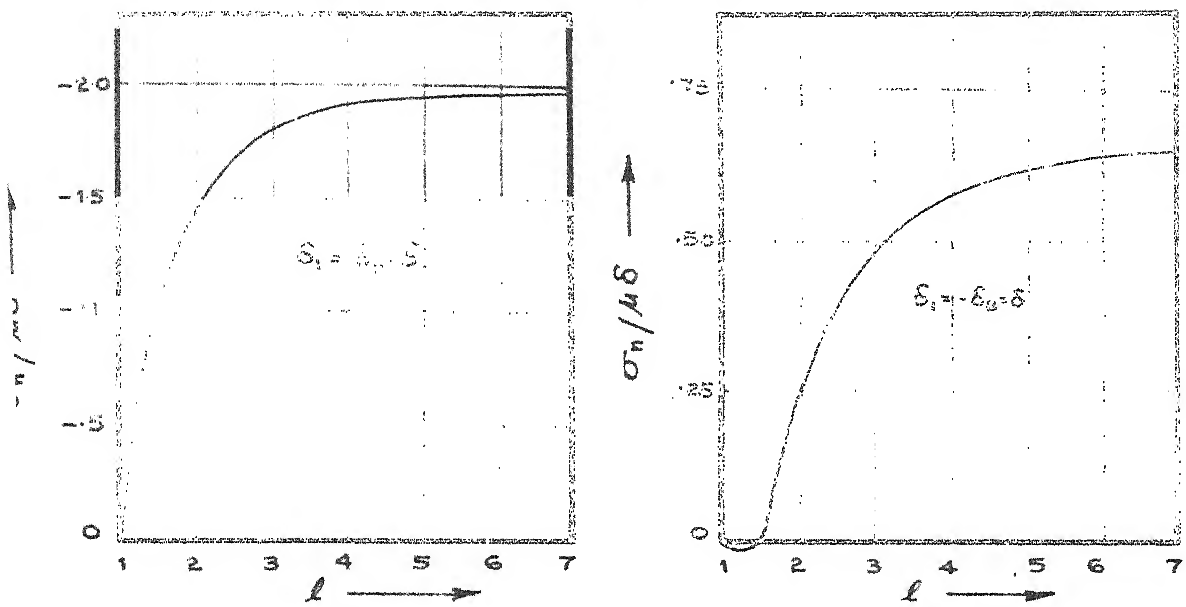
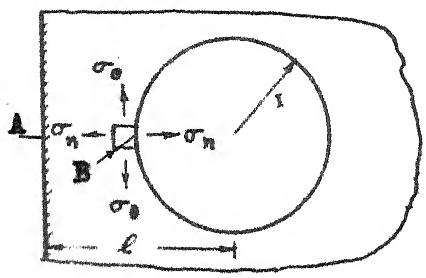


Figure 10. Inclusion in the presence of a hole; coordinate system.

Figure 11. Normal stress at the point B.





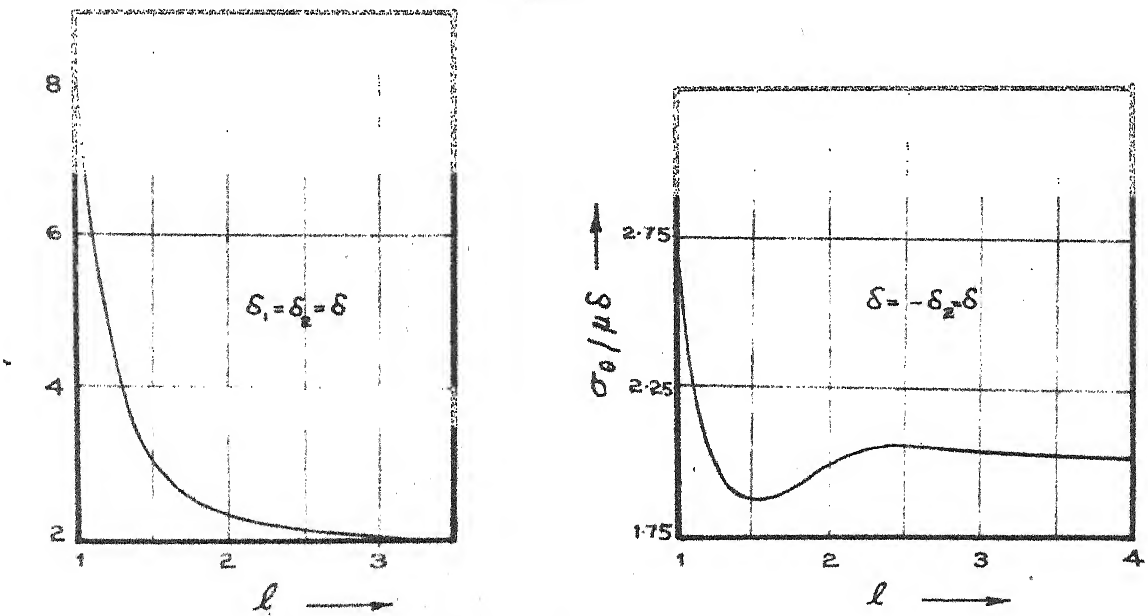


Figure 12. Hoop stress at the point B (matrix).

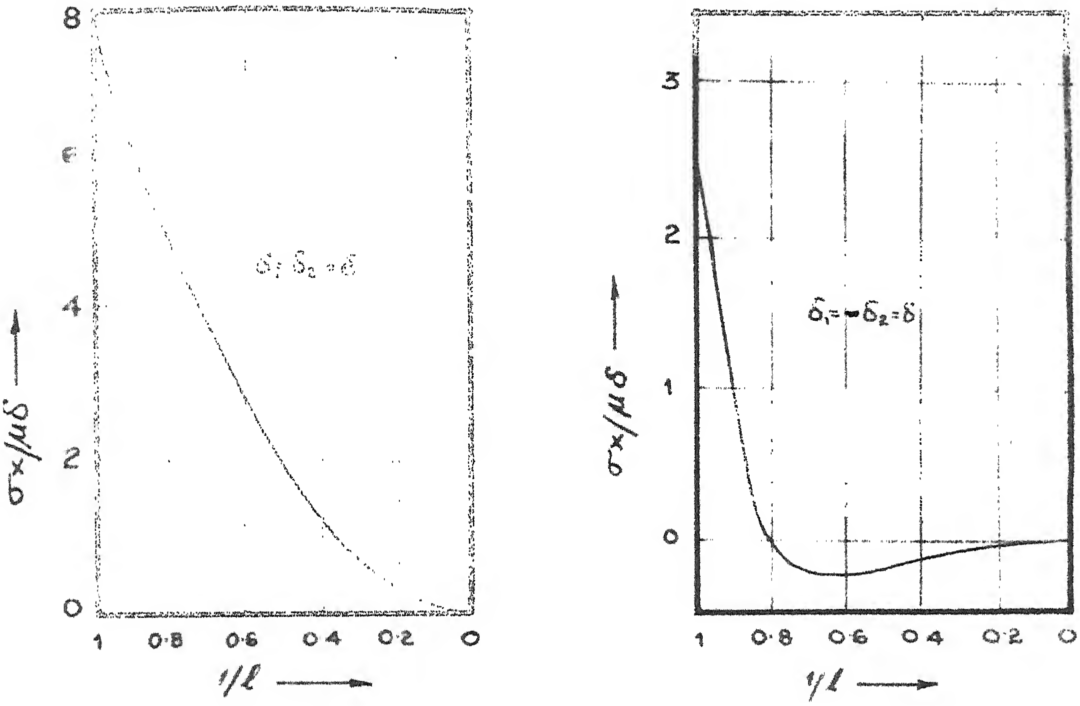


Figure 13. Hoop stress $\sigma_x/\mu\delta$ at the point A(Fig. 9)

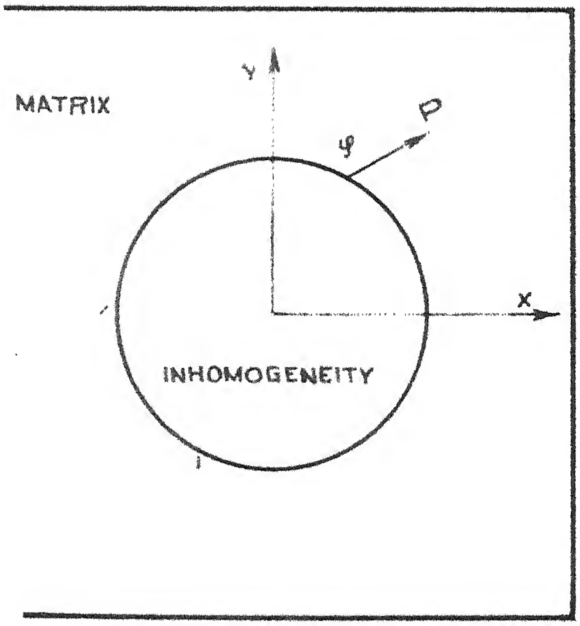


Figure 14. Inhomogeneity and point-force.

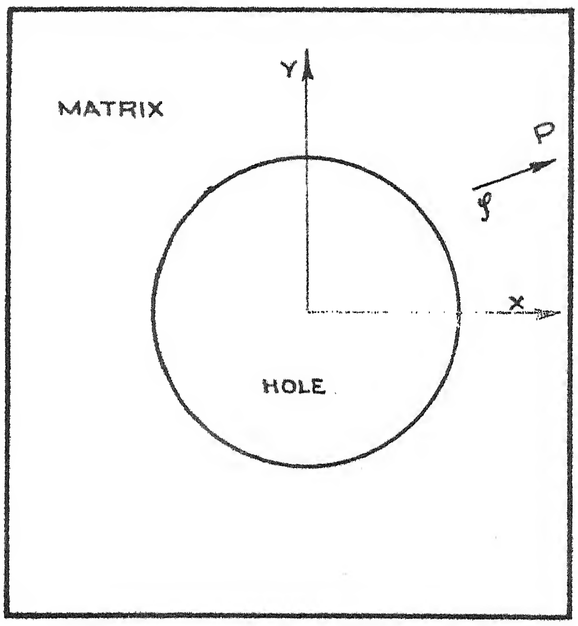


Figure 15. Hole and the point-force.

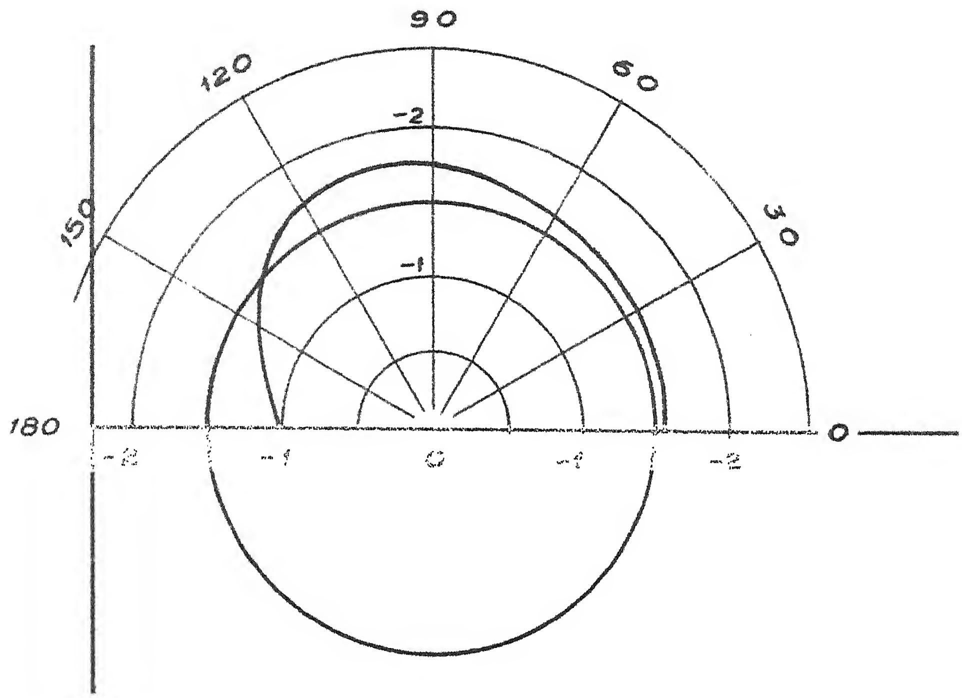


Figure 16. Normal stress $\sigma_n / \mu\delta$; $\delta_1 = \delta_2 = \delta$

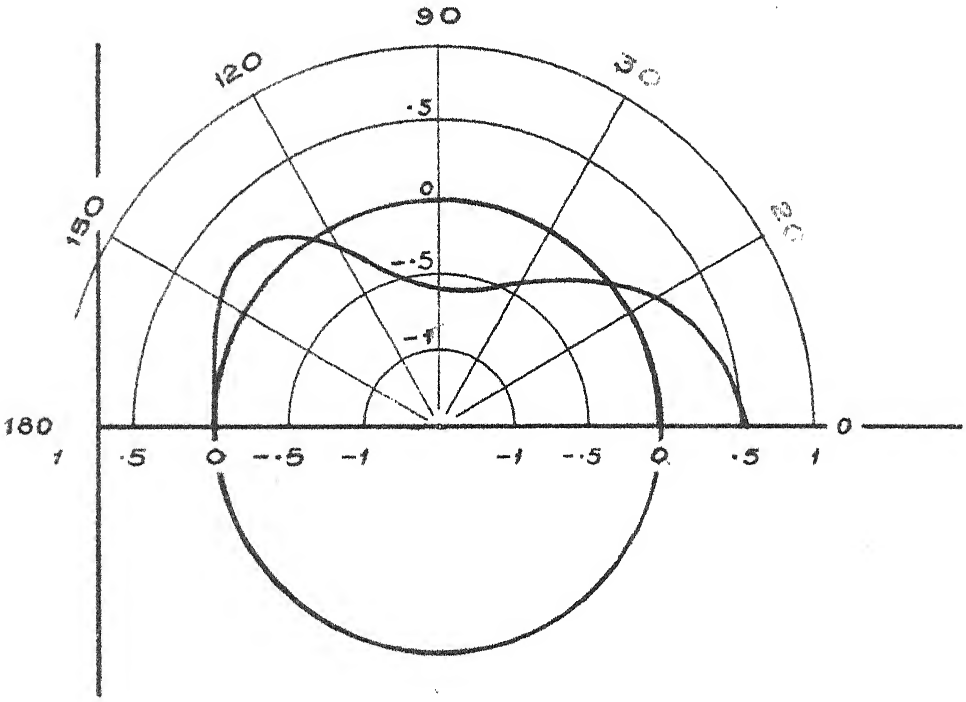


Figure 17. Normal stress $\sigma_n / \mu\delta$; $\delta_1 = -\delta_2 = \delta$

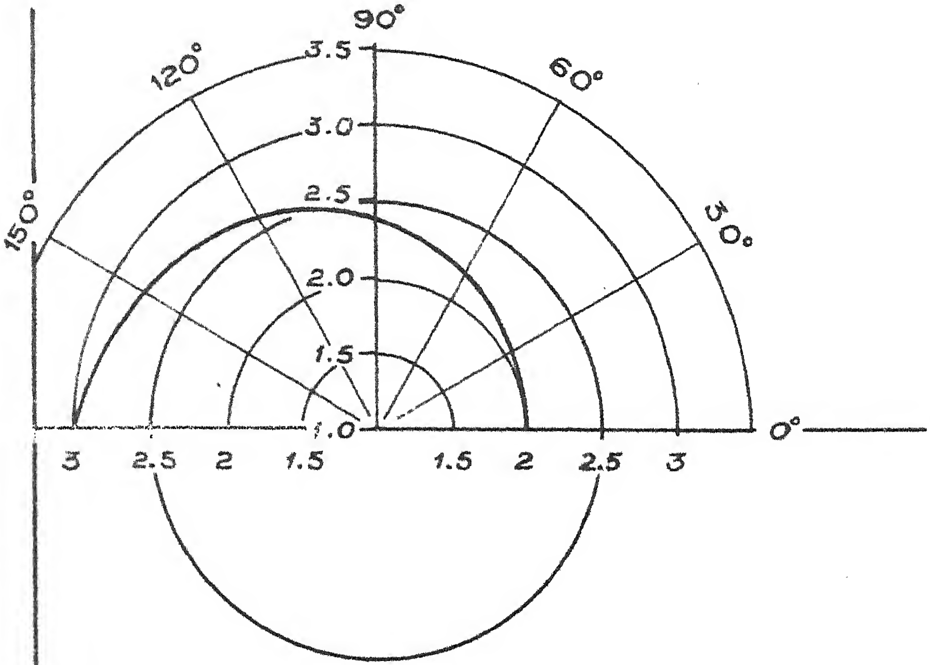


Figure 18. Hoop stress out-side, $\delta_1 = \delta_2 = \delta$

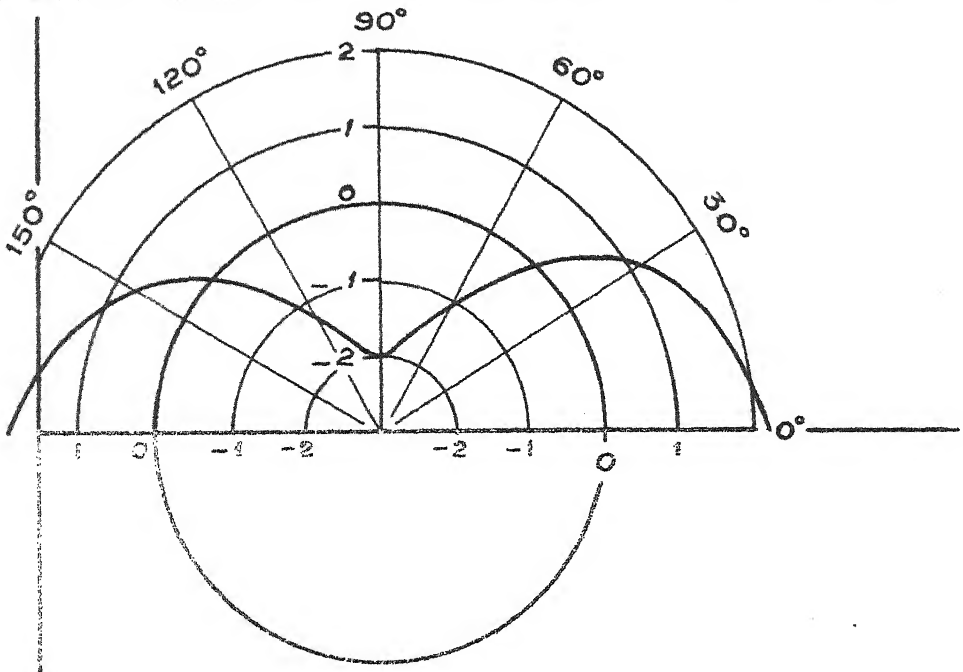


Figure 19. Hoop stress outside, $\delta_1 = -\delta_2 = \delta$

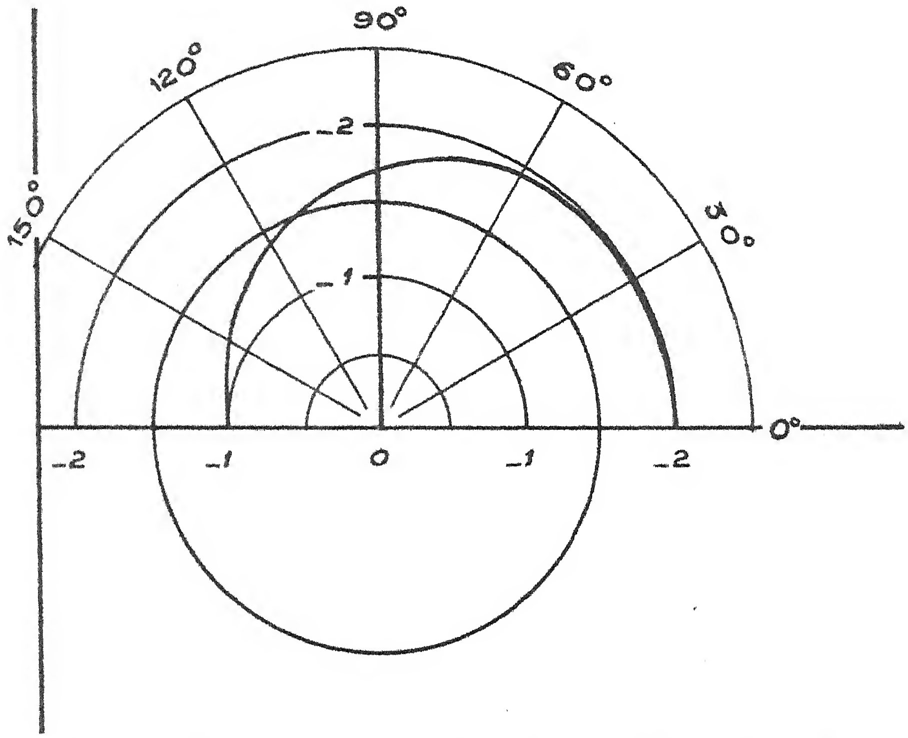


Figure 20. Hoop stress inside $(\sigma_s)_i/\mu\delta$; $\delta_1 = \delta_2 = \cdot\delta$.

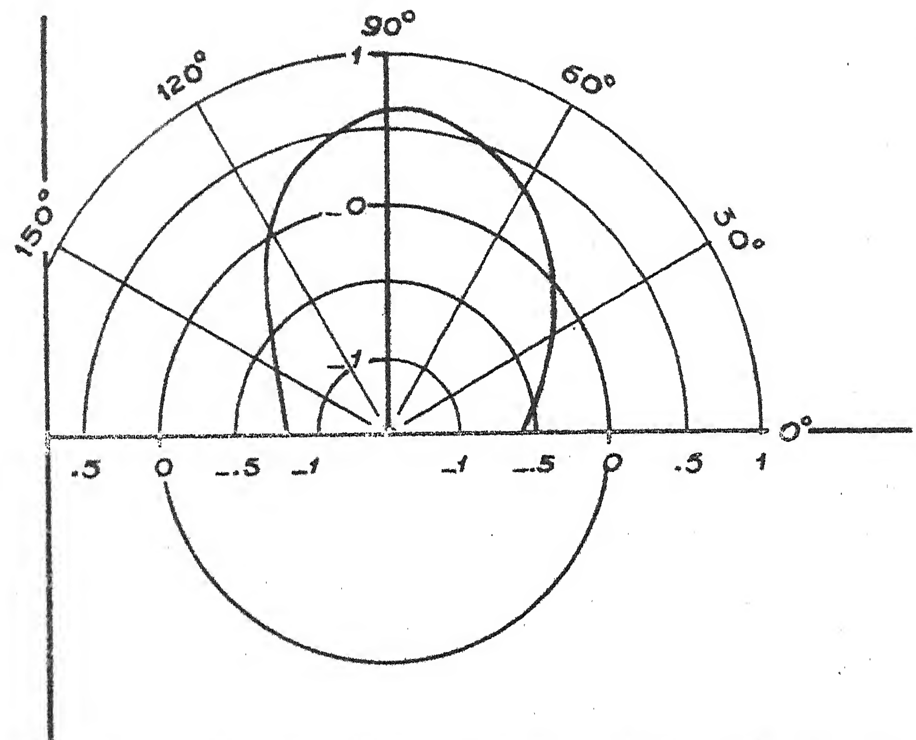


Figure 21. Hoop stress inside $(\sigma_s)_i/\mu\delta$; $\delta_1 = -\delta_2 = \delta$.

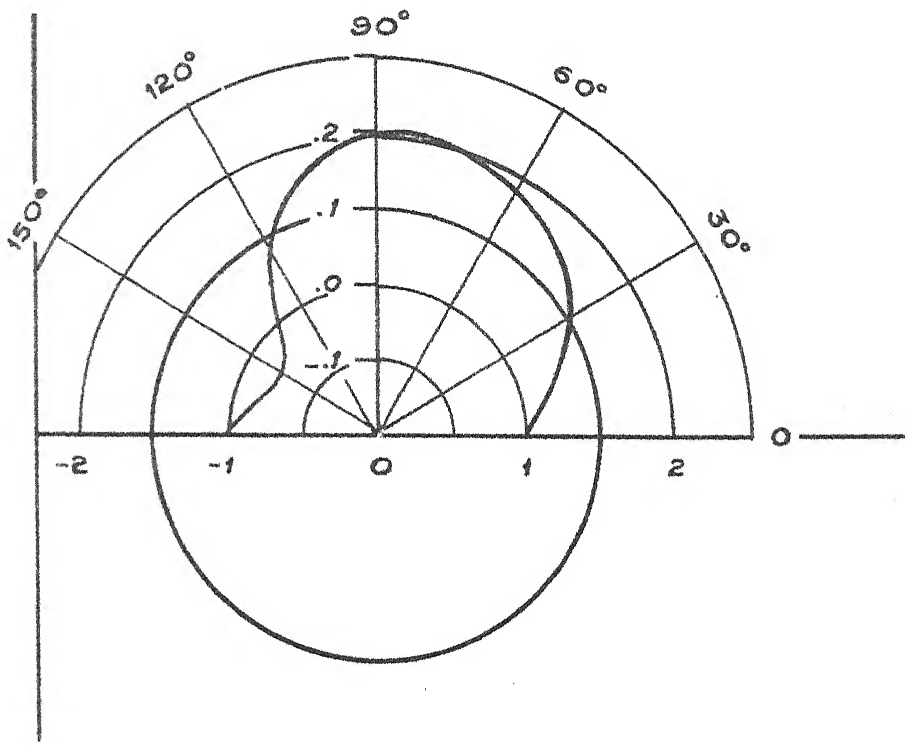


Figure 22. Tangential stress $\tau_{ns}/\mu\delta$; $\delta_1 = \delta_2 = \delta$.

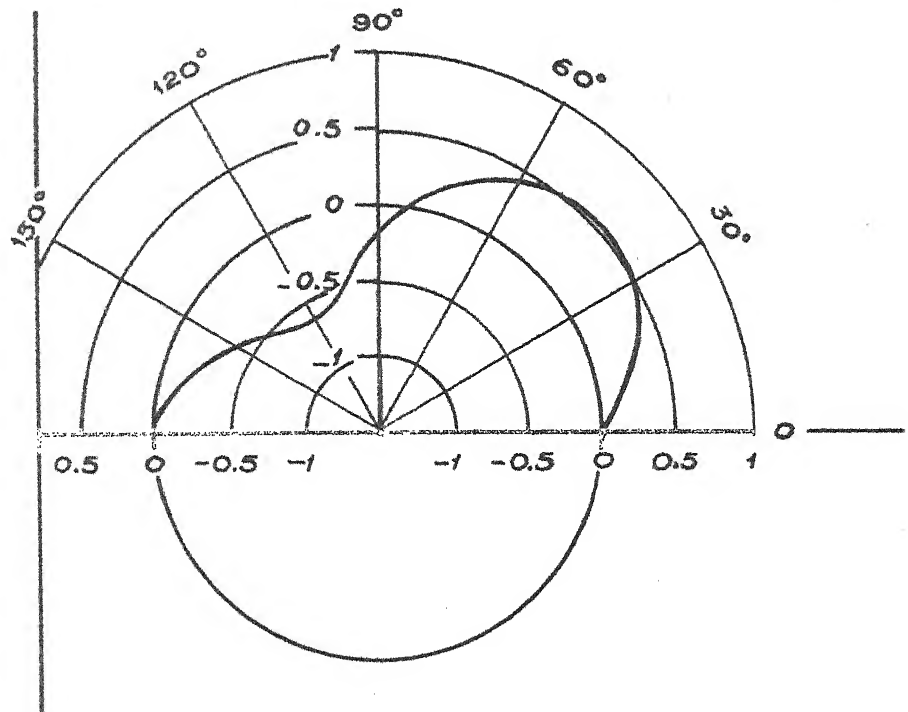


Figure 23. Tangential stress $\tau_{ns}/\mu\delta$; $\delta_1 = -\delta_2 = \delta$

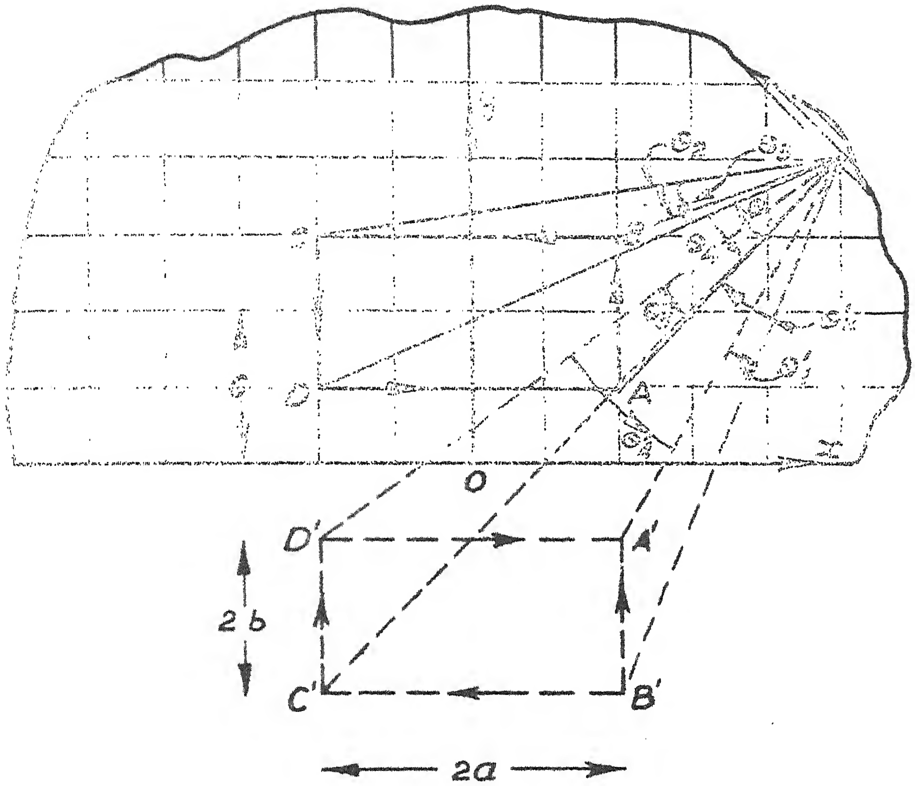


Figure 24. Rectangular inclusion in semi-infinite medium and coordinate system.

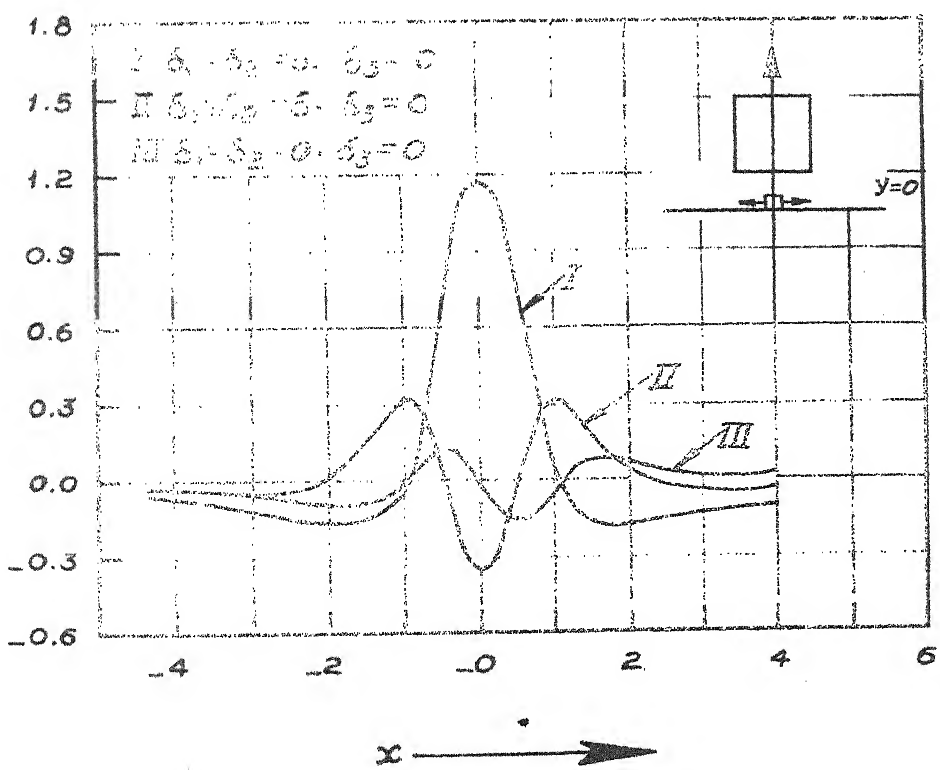


Figure 25. Hoop

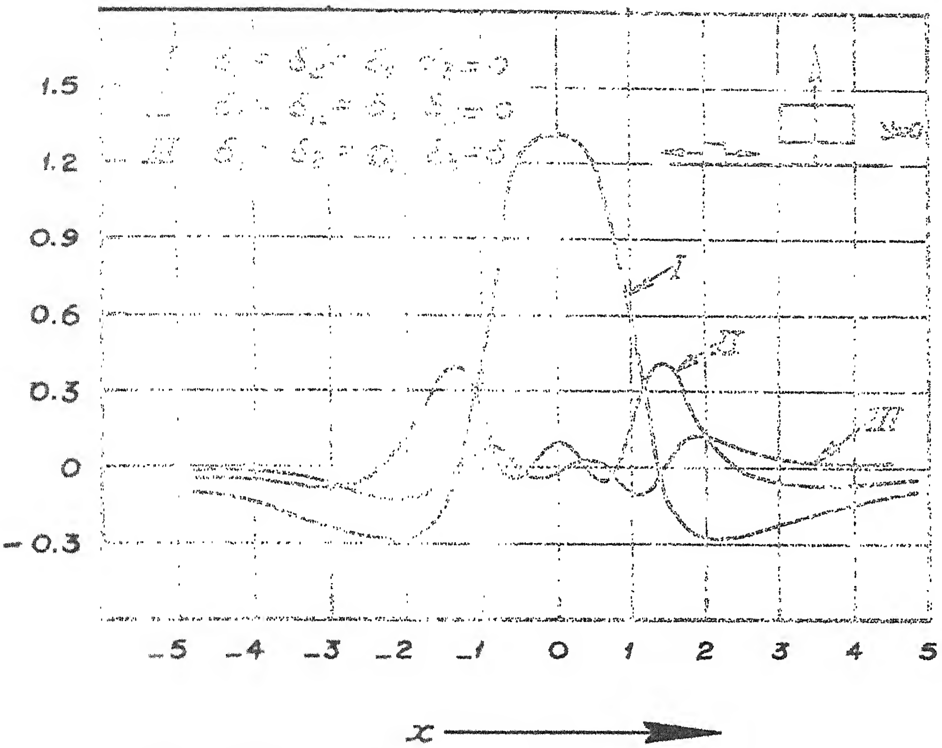


Figure 26. Hoop stress along the leading edge for the case of 1×2 rectangle, c = 1.

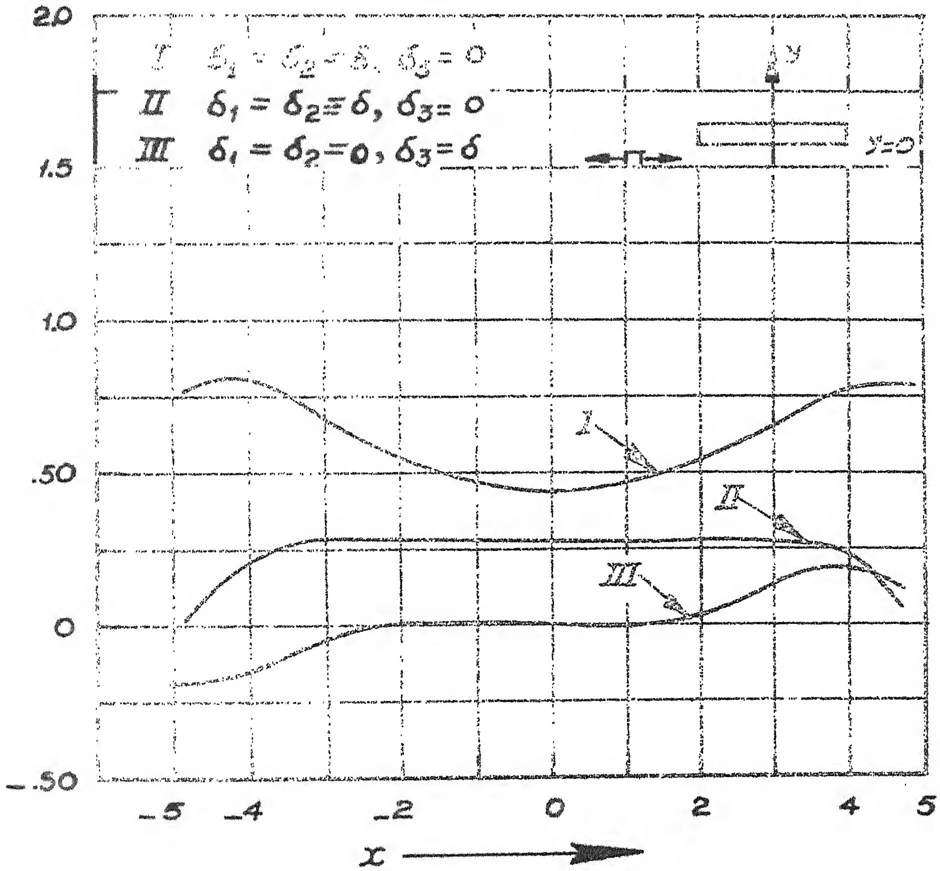


Figure 27. Hoop stress along the leading edge

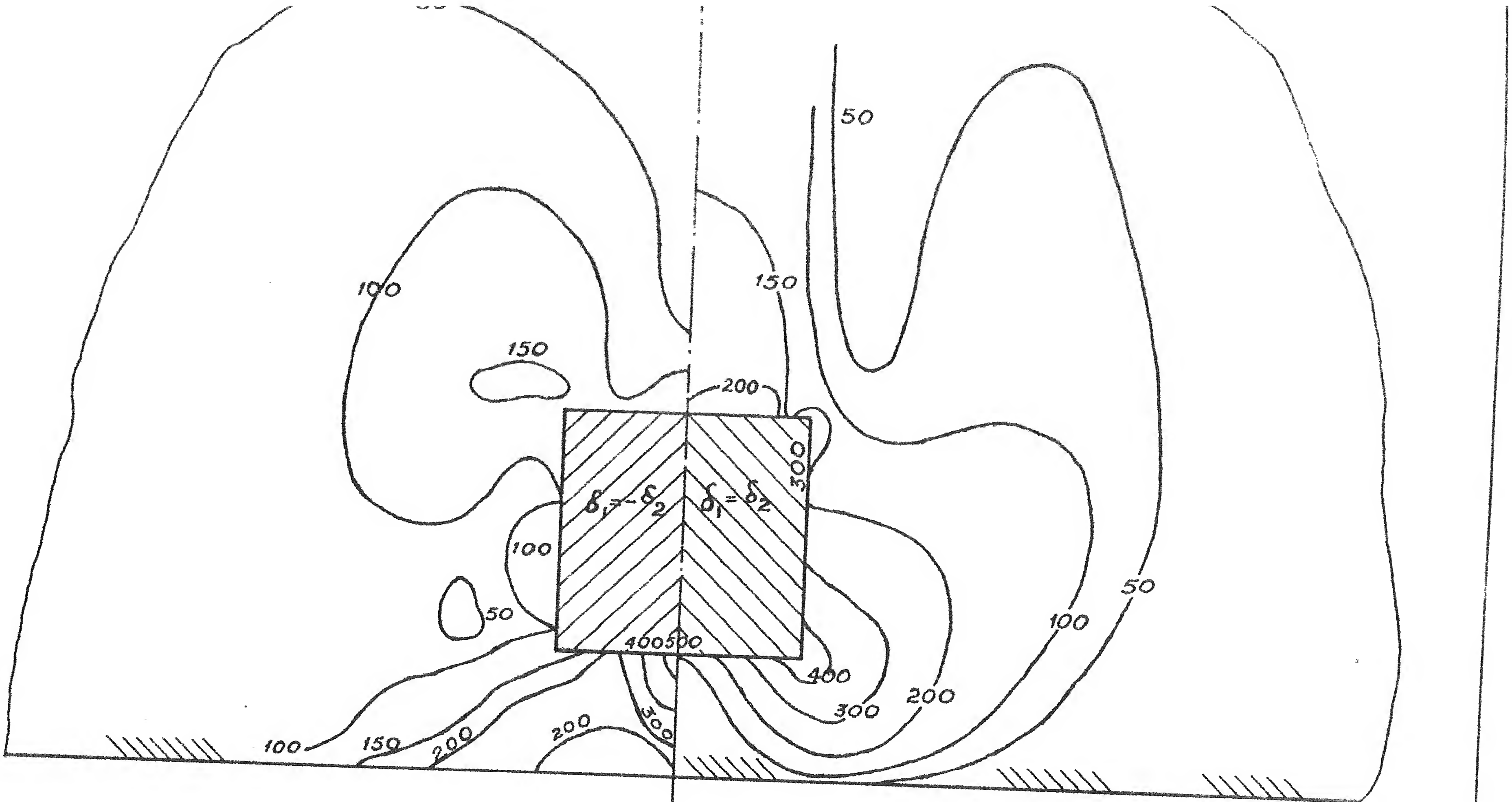


Figure 28. Lines of maximum shearing stress around a square.

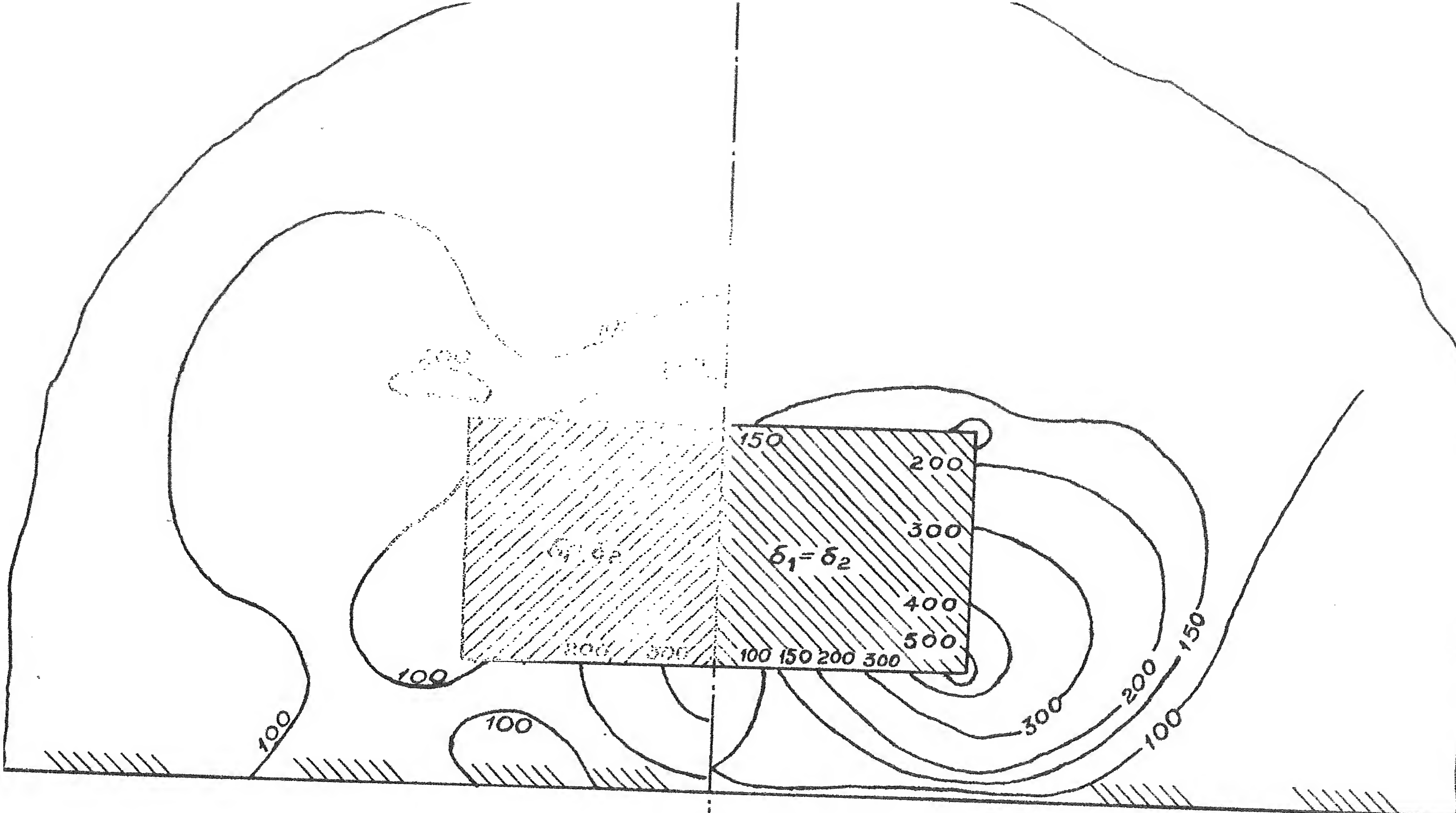


Figure 29. Lines of maximum stress ar-ound the 1 x 2 rectangle ; c = 1.

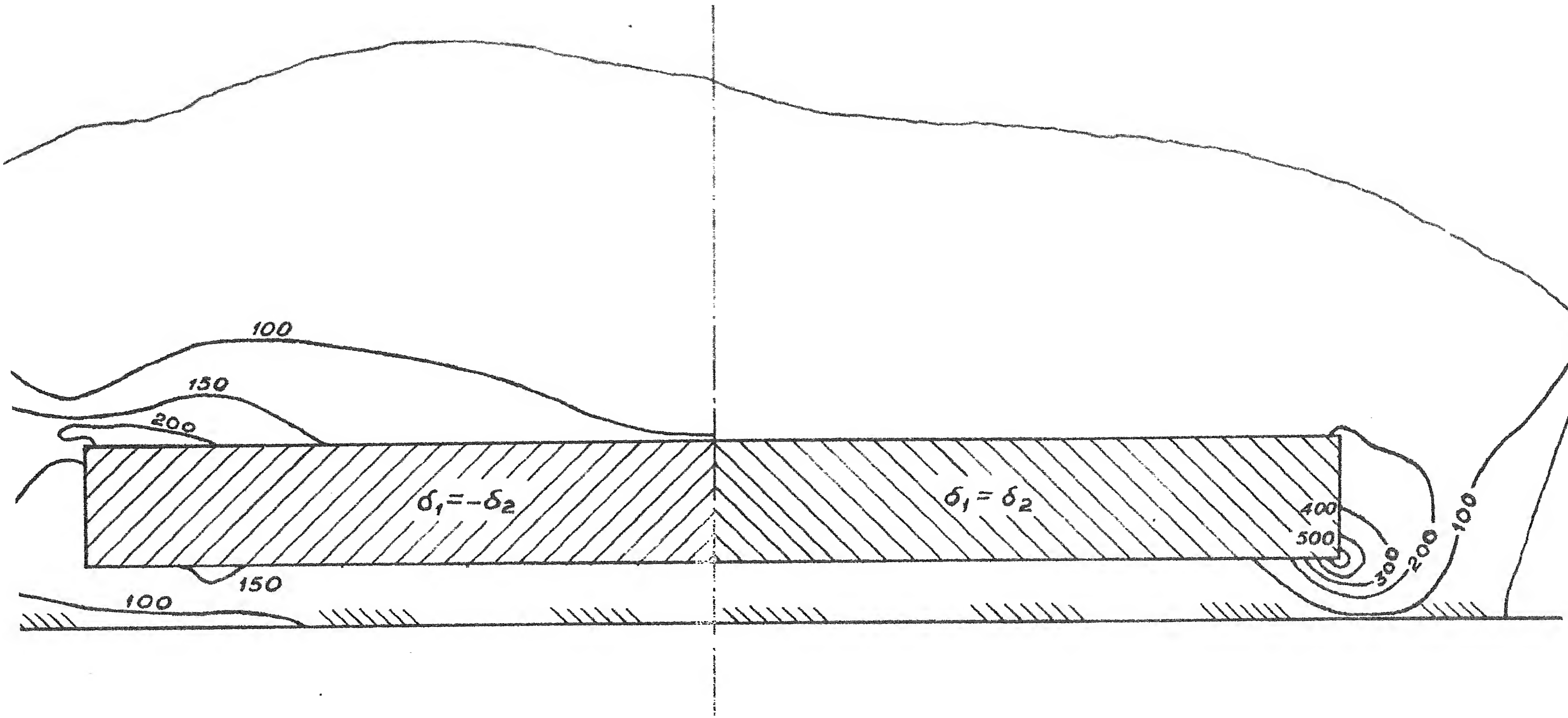


Figure 30. Lines of maximum shearing stress around the 1×10 rectangle; $c = 1$.

μ/μ_h	0	1/3	1	3	∞
	I	II	III	IV	V

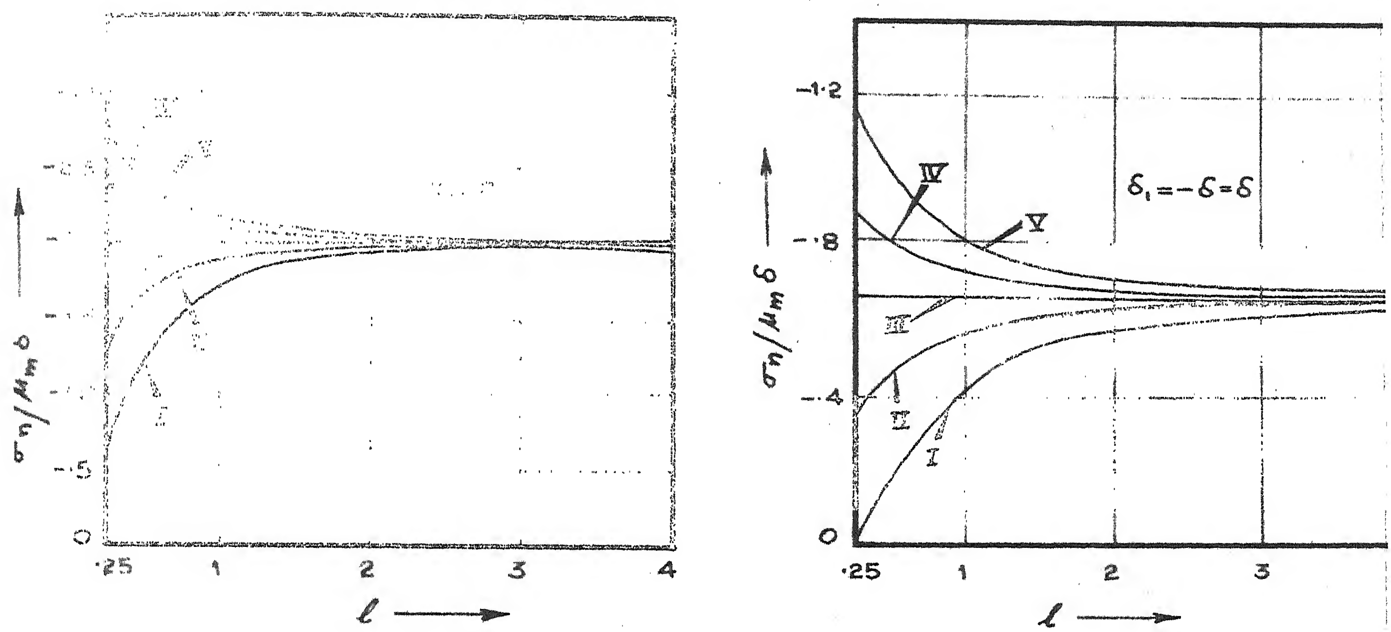
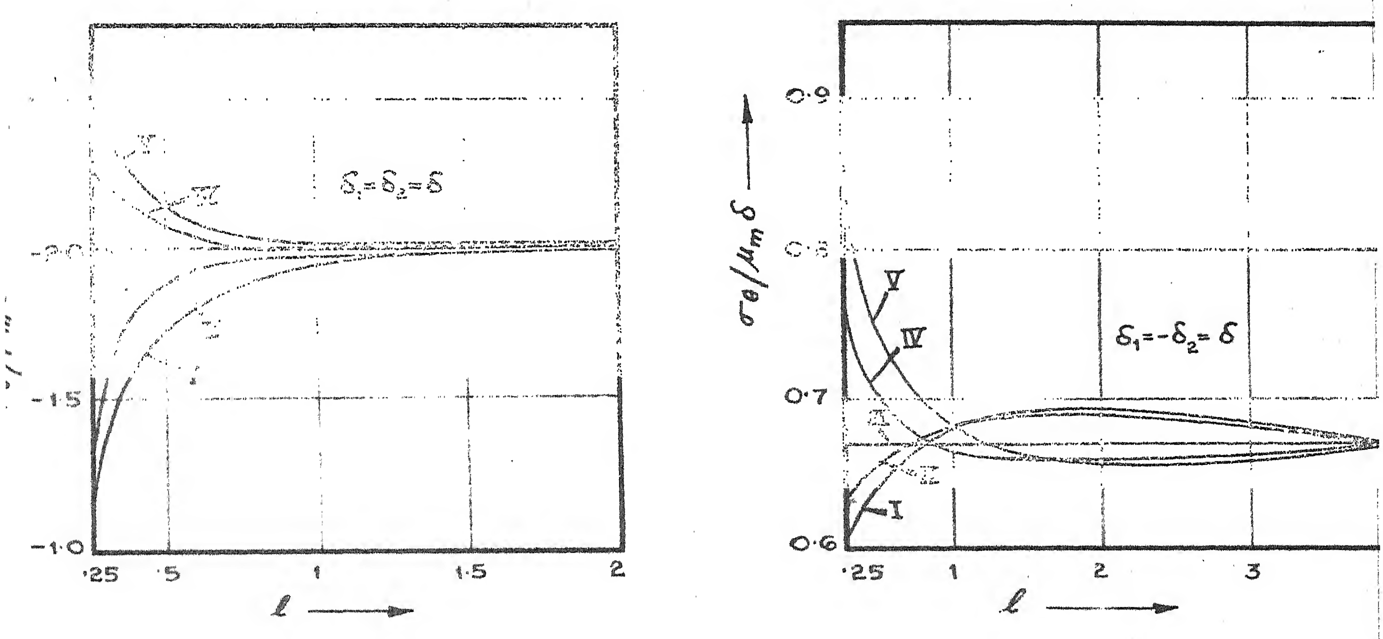


Figure 31. Normal stress at the point B (Fig. 10), for various values of β and ℓ.



μ_{it}/μ_m	0	·5	1	4	∞
	I	II	III	IV	V

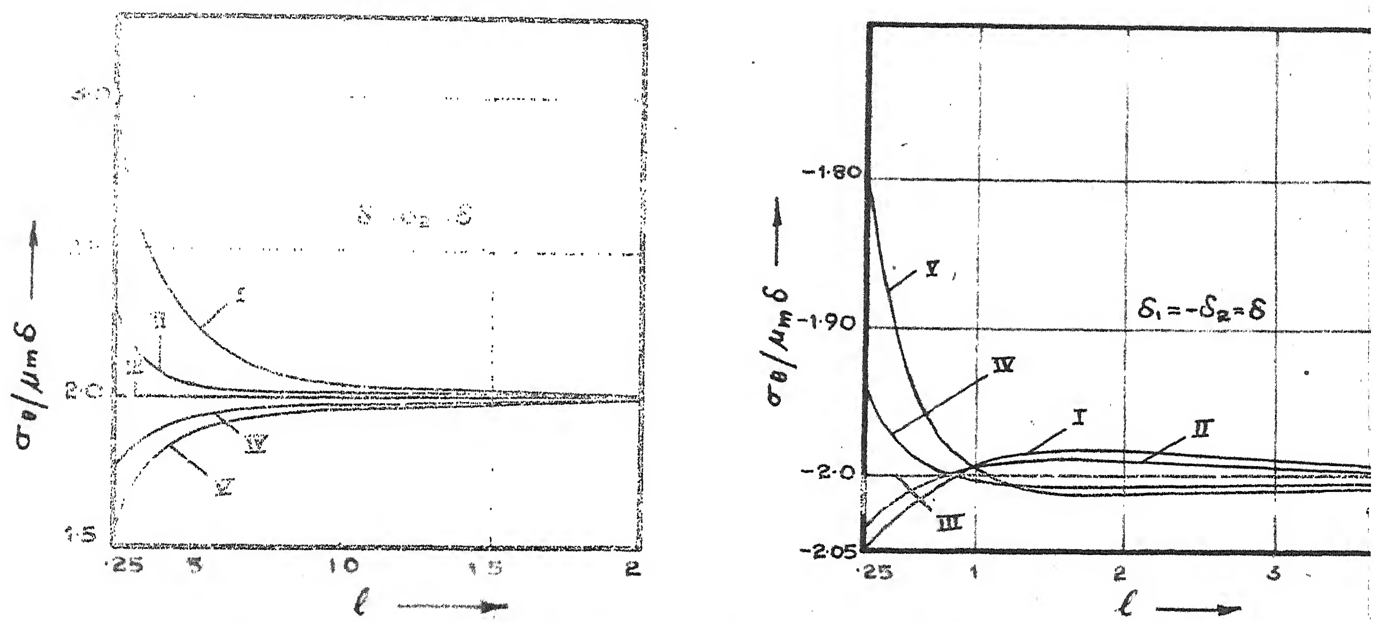
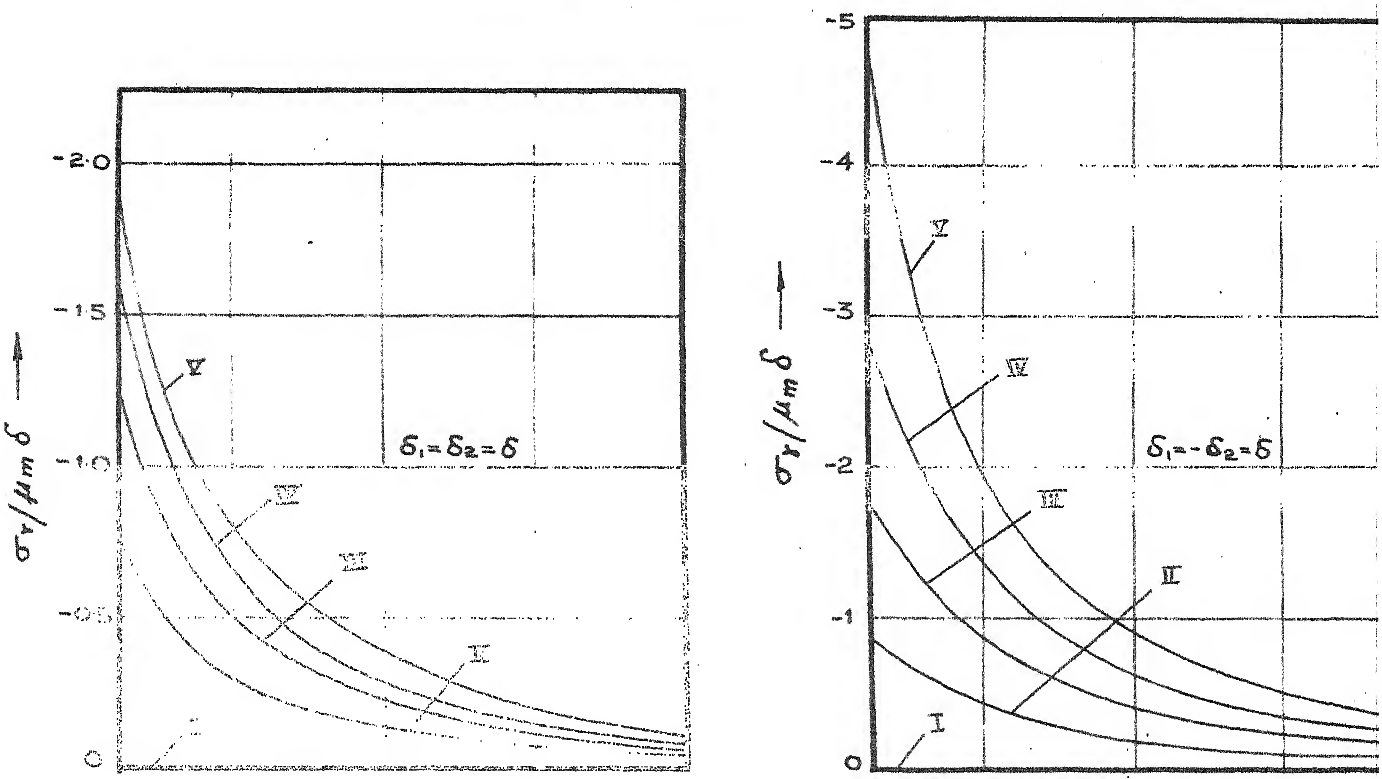


Figure 33. Hoop stress at the point B (Fig. 10), for various values of $\frac{\mu_{it}}{\mu_m}$ and ℓ .



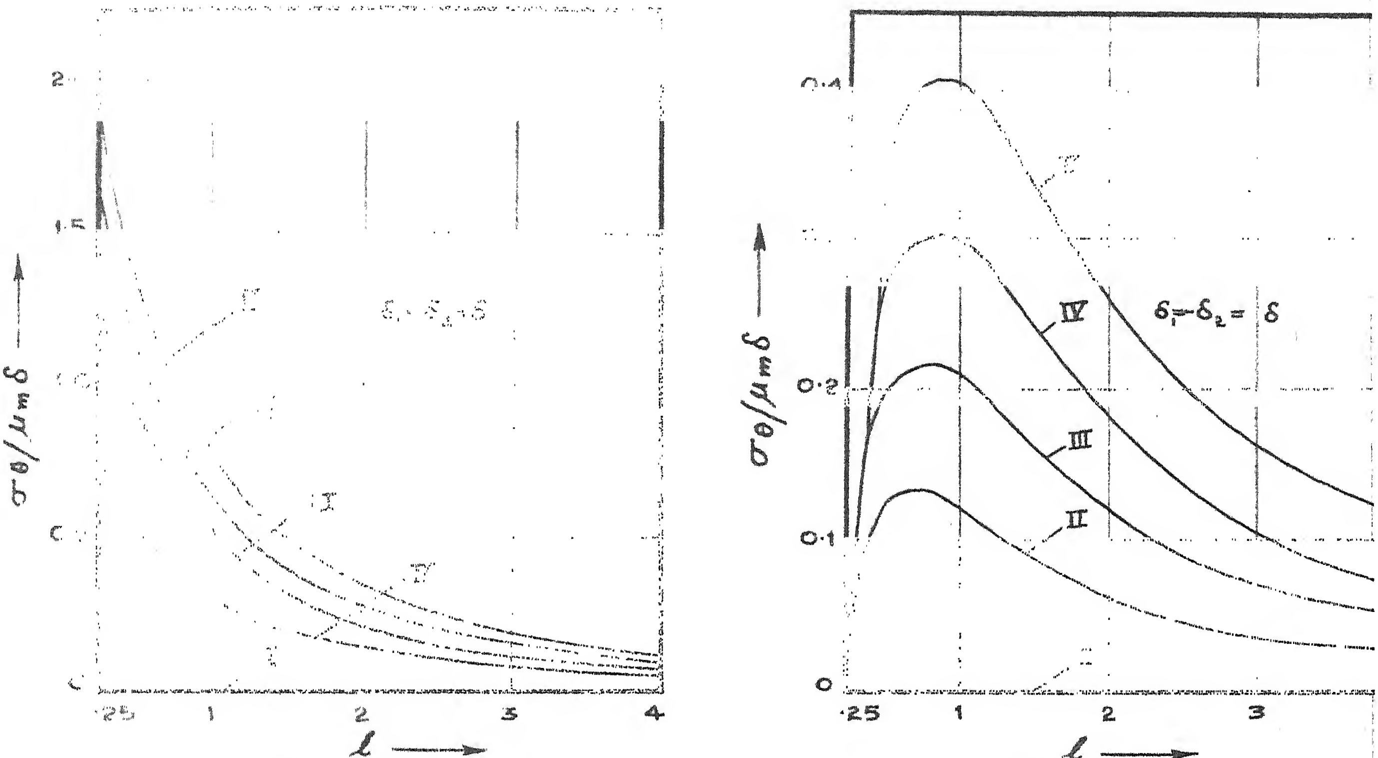
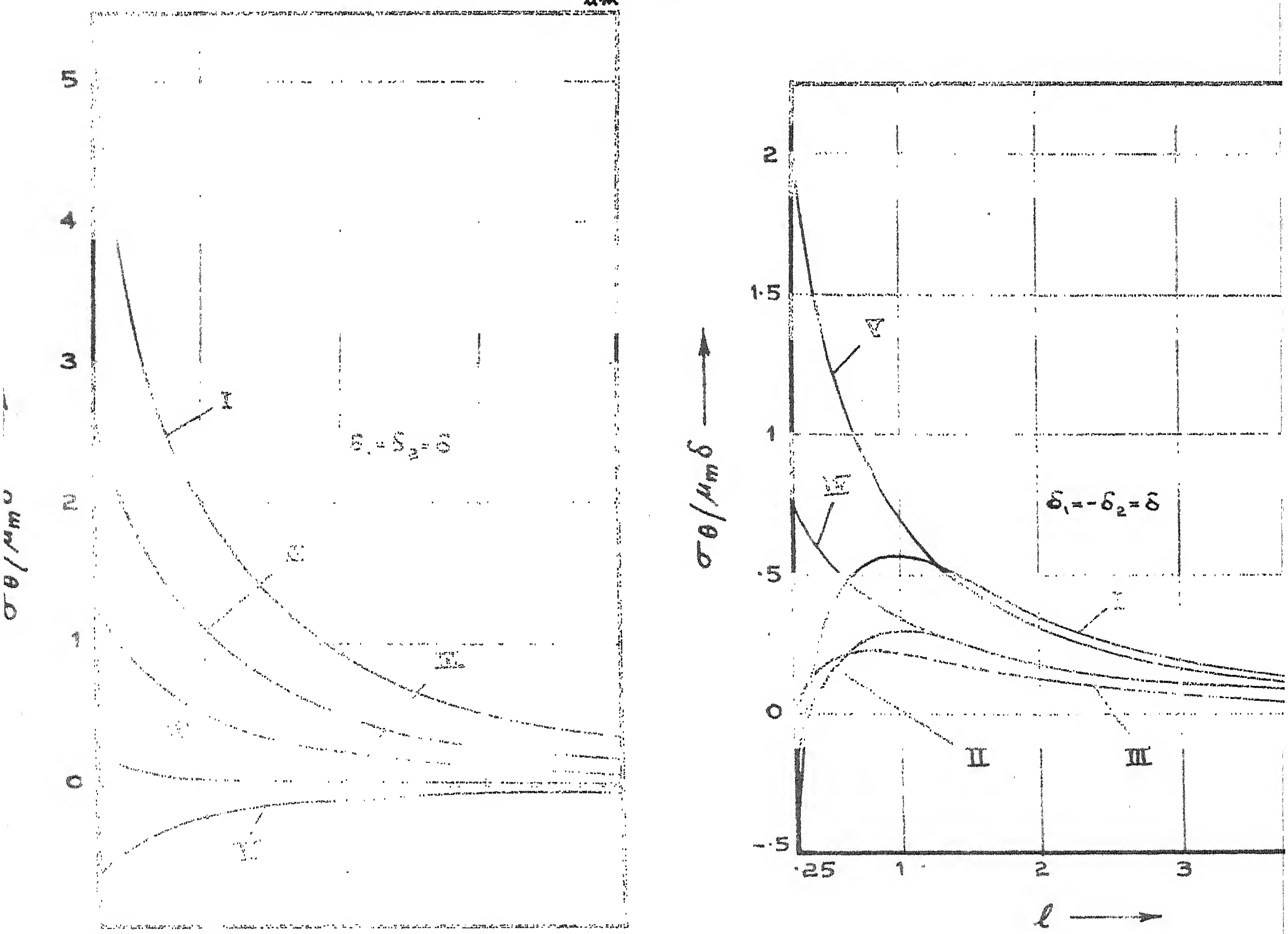


Figure 35. Hoop stress inside at the point A (Fig. 10), for various $\frac{u_{ih}}{u_m}$ and ℓ .



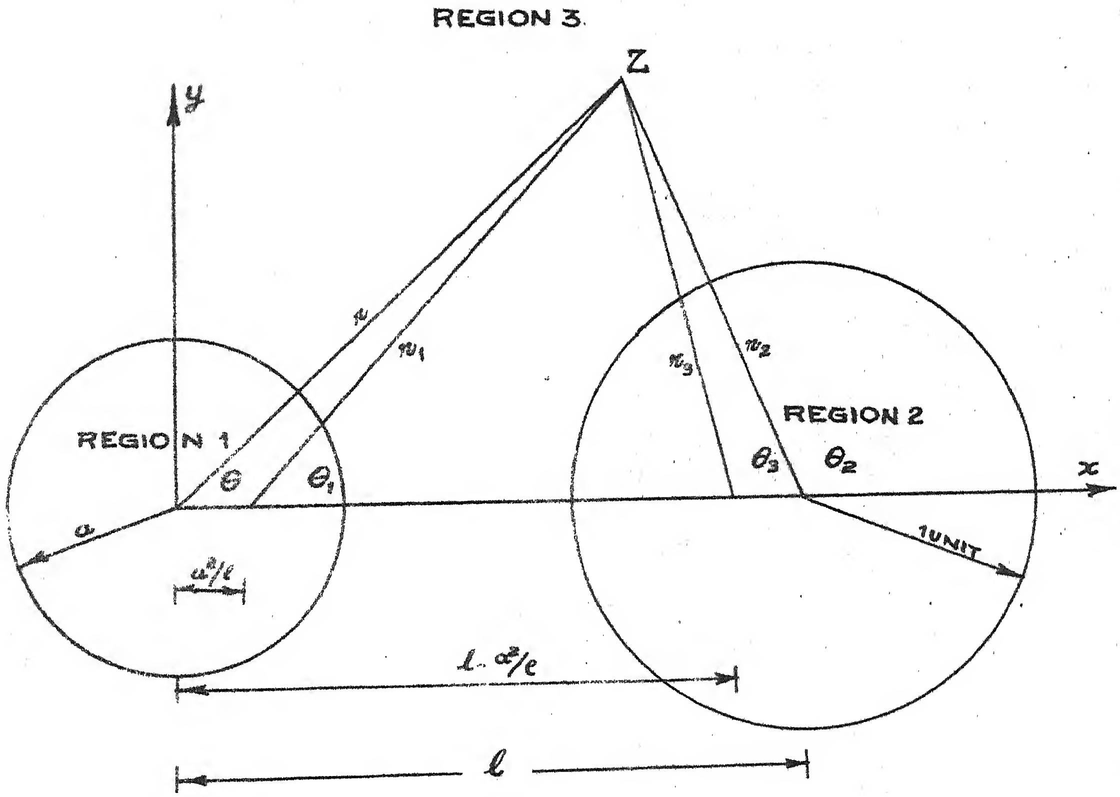


FIG 37 COORDINATE SYSTEM AND CONFIGURATION.

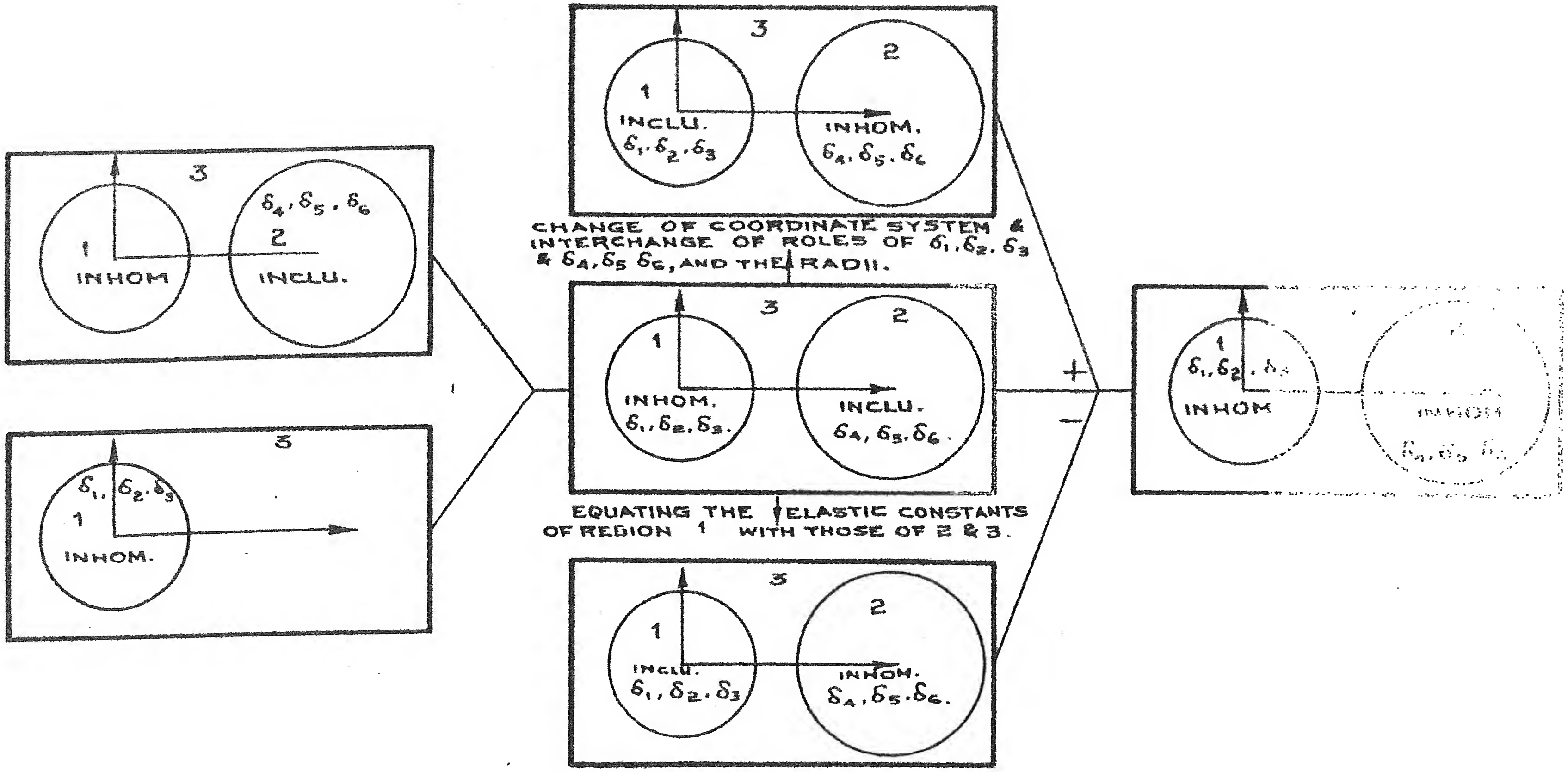


FIG. 38 FLOW CHART OF THE PROCESS OF SUPERPOSITION.

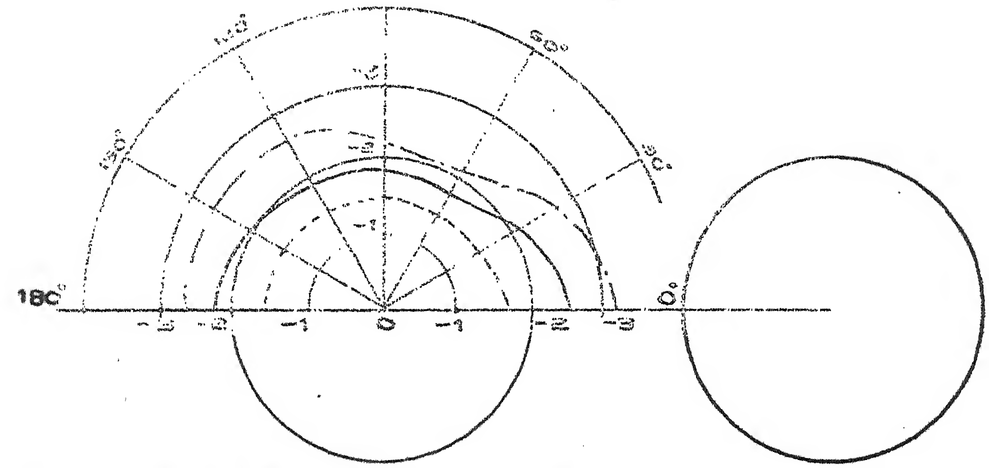


FIG. 39 NORMAL STRESS σ_n/μ_m . $\delta_1=\delta_2$. – – – – –, $\beta=0{\cdot}5$; ———, $\beta=1$; —·—·—, $\beta=2$.

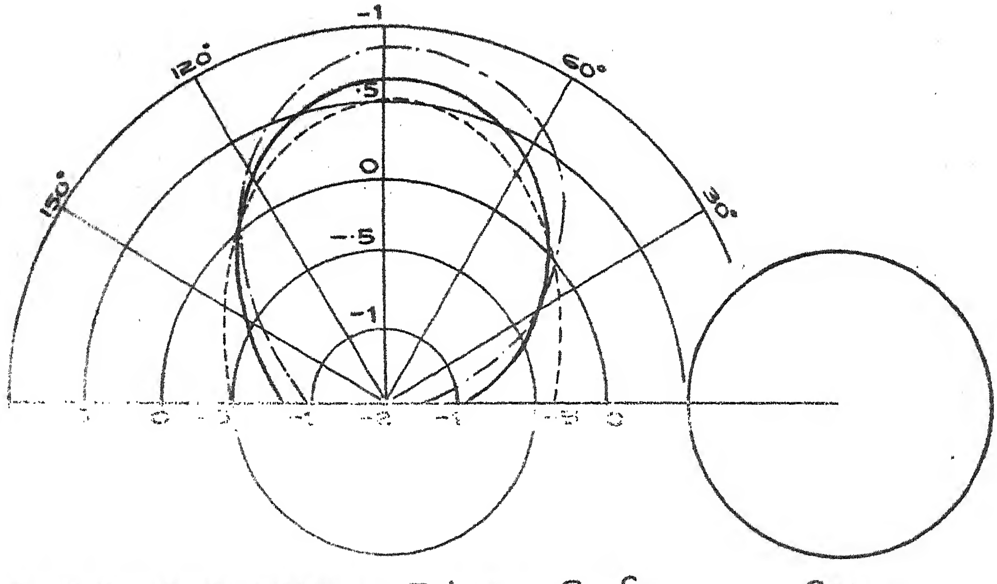


FIG. 40. NORMAL STRESS σ_n/μ_m . $\delta_1=-\delta_2$. – – – – – –, $\beta=0{\cdot}5$;

———, $\beta=1$; —·—·—·—, $\beta=2$.

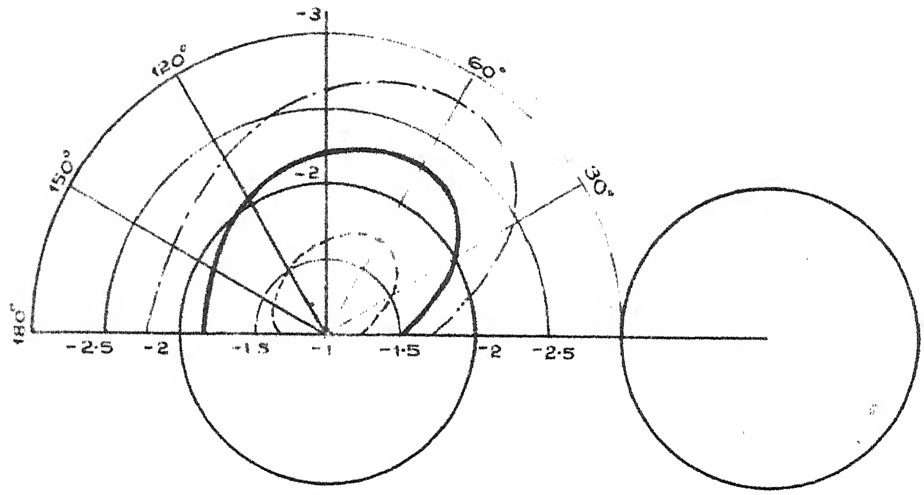


FIG. 41. HOOP STRESS σ_0/μ_m INSIDE. $\delta_1 = \delta_2$. ------ , $\beta = 0{\cdot}5$;

——— $\beta = 1$; —·—·— $\beta = 2$;

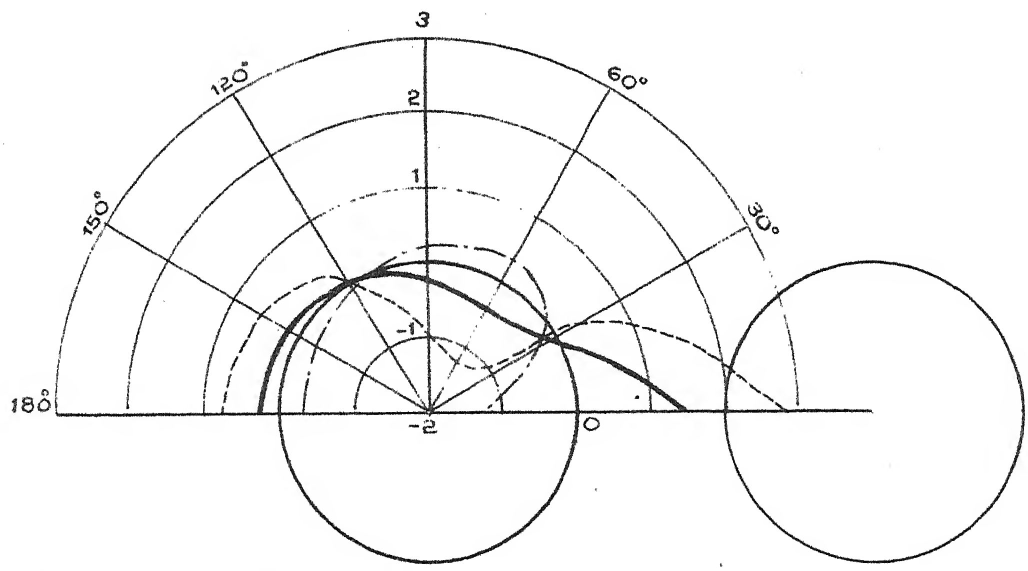


FIG. 42 HOOP STRESS σ_0/μ_m OUTSIDE. $\delta_1 = \delta_2$. ------ , $\beta = 0{\cdot}5$,

——— , $\beta = 1$; —·—·— , $\beta = 2$.

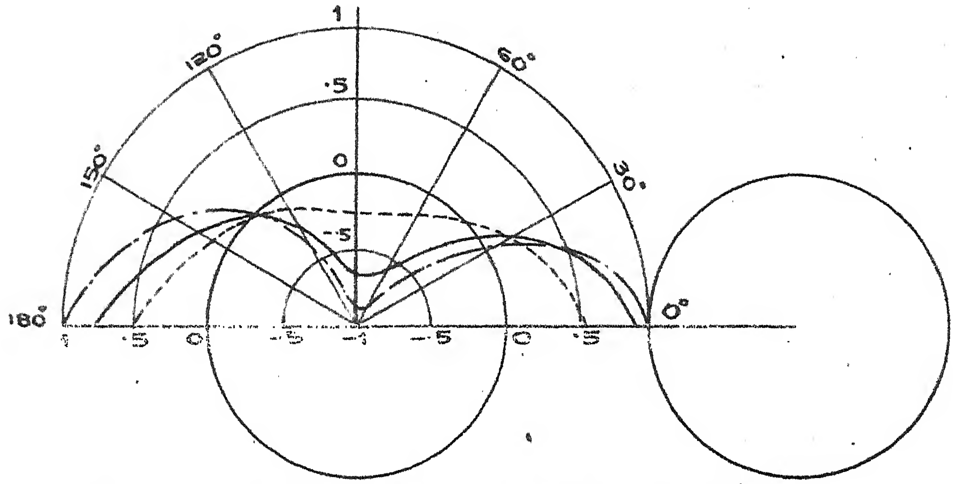


FIG. 43. HOOP STRESS σ_θ/μ_m INSIDE. $\delta_1 = -\delta_2$. ------ , $\beta = 0{\cdot}5$
———, $\beta = 1$. ; —·—·— , $\beta = 2$.

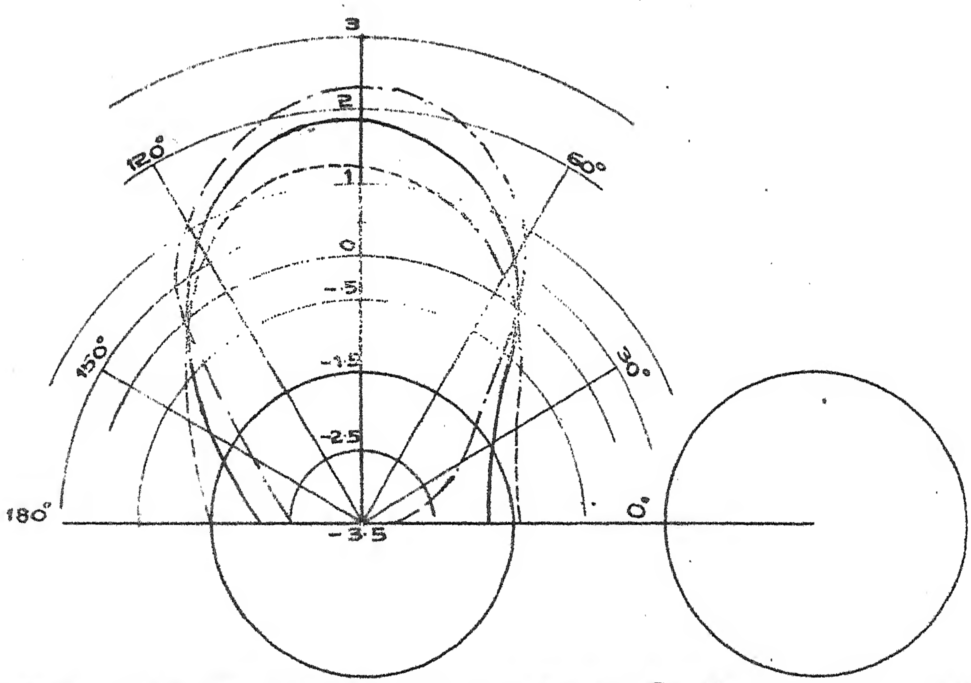


FIG. 44 HOOP STRESS σ_θ/μ OUTSIDE. $\delta_1 = -\delta_2$. ------ $\beta = 0{\cdot}5$
———, $\beta = 1$; —·—·— , $\beta = 2$.

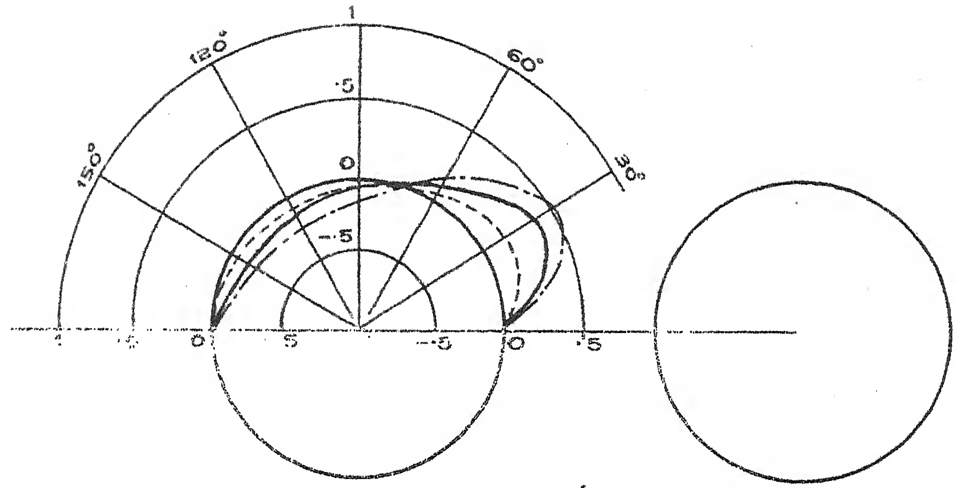


FIG. 45 TANGENTIAL STRESS $\tau_{r\theta}/\mu$ $\delta_1 = \delta_2$. ------, $\beta = 0{\cdot}5$;
——, $\beta = 1$; —·—·—, $\beta = 2$.

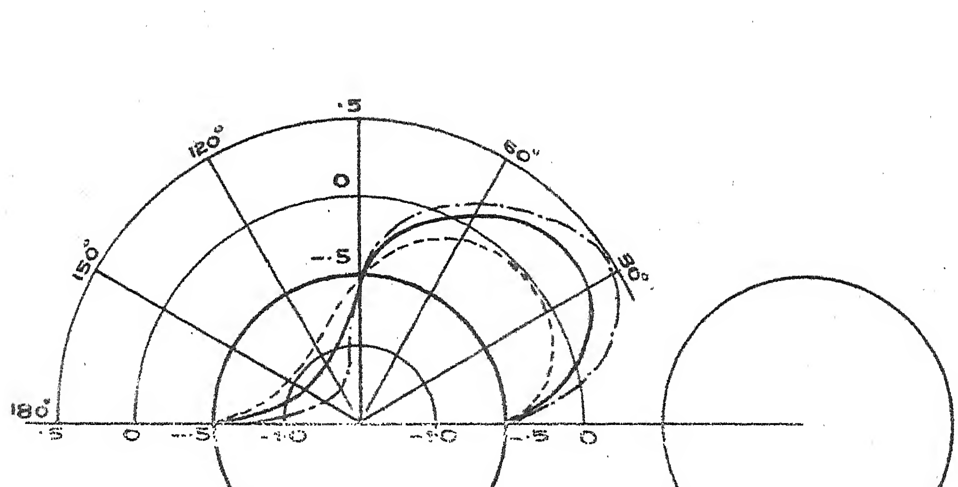


FIG 46 . TANGENTIAL STRESS $\tau_{r\theta}/\mu$ $\delta_1 = -\delta_2$. ------, $\beta = 0{\cdot}5$;
—— $\beta = 1$; —·—·—, $\beta = 2$.

NORMAL STRESS , $\sigma_n/\mu\delta$

L=6.0

θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$
0	-1.9534185	.63251183
30	-1.9634450	.31431533
60	-1.9804145	-.32924347
90	-1.9858317	-.65823219
120	-1.9781953	-.33661209
150	-1.9673999	.31675344
180	-1.9626764	.64472356

TABLE 1 : Stresses along the boundary of the circular inclusion in a semi-infinite medium of Chapter IV ; (Plane stress case; Poisson's ratio = 1/3).

180	-1.5625000	.52083336

L=1

Θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$
0	.0000000	0.000000
30	-.2222479	-.05545433
60	-1.3333338	-.00000037
90	-1.4080001	-.56917343
120	-1.2944609	-.39792990
150	-1.2124390	.20816741
180	-1.1851852	.52674899

($\tau_{ms}/\mu\delta$) TANGENTIAL STRESS

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L=6.0	
	.00000004	-.00000168
30	-.01284196	-.55771959
60	-.00925718	-.56407905
90	.00450301	-.00581983
120	.0133188	.56129003
150	.01068989	.56512355
180	.00000002	.00000173
	L=2.5	
	.00000027	-.00000120
30	-.06789536	-.45116158
60	-.01739338	-.54425026
90	.05461993	-.06104342
120	.07575651	.50251391
150	.04956887	.53289006
180	.00000013	.00000165
	L=2.0	
	.00000039	-.00000081
30	-.0891454	-.38811610
60	.00157684	-.55599883
90	.09769997	-.09925505
120	.11333795	.47633662
150	.06971236	.52143959
180	.00000016	.00000164
	L=1.5	
	.00000031	-.00000044
30	-.05951186	-.30772521
60	.08584518	-.60180876
90	.19199998	-.16095999
102	.18053698	.44354766
150	.10275957	.50916007
180	.00000022	.00000163
	L=1	
	.00001064	.00000347
30	.81880762	-.21755106
60	.38490006	-.64150040
90	.38399996	-.22528001
120	.29288318	.41957220
150	.15342512	.50334140
180	.00000029	.00000161

($\sigma_3 / \mu\delta$) HOOP STRESS INSIDE

θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$
	L=6.0	
0	-1.9804658	-.65801103
30	-1.9724100	-.33823443
60	-1.9601136	.30883017
90	-1.9597569	.64120849
120	-1.9713362	.32191159
150	-1.9844903	-.33020824
180	-1.9899864	-.65779496
	L=2.5	
0	-1.8125000	-.64062499
30	-1.7730650	-.41178555
60	-1.7502429	.18055353
90	-1.8038235	.57098320
120	-1.8830853	.31490488
150	-1.9420516	-.30446837
180	-1.9629629	-.62294242
	L=2.0	
0	-1.6296297	-.65843613
30	-1.5956136	-.48436951
60	-1.6149295	.11363576
90	-1.7268473	.55625716
120	-1.8464527	.326108
150	-1.9253416	-.26657344
180	-1.9520000	-.60415999
	L=1.5	
0	-1.0000000	-.79166674
30	-1.1476636	-.62896080
60	-1.3848397	.04309775
90	-1.6240000	.56858643
120	-1.8029129	.36008898
150	-1.9049591	-.25347925
180	-1.9375000	-.57291676
	L=1	
0	4.0000000	0.000000
30	-.2647276	-.86854237
60	-1.3333338	.00000067
90	-1.6320000	.62890653
120	-1.8075801	.42458550
150	-1.8982268	-.20196991
180	-1.9259260	-.52674899

($\sigma_3/\mu\delta$) HOOP STRESS OUTSIDE

Θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$
	L=6.0	
0	2.0195342	2.00865560
30	2.0275900	.99509490
60	2.0398864	1.02450490
90	2.0402431	2.02545820
120	2.0286638	1.01142390
150	2.0155097	1.00312070
180	2.0100136	2.00887170
	L=2.5	
0	2.1875000	2.02604170
30	2.2269350	.92154380
60	2.2497571	1.15278150
90	2.1961765	2.09568340
120	2.1169147	1.01843060
150	2.0579484	1.02886060
180	2.0370371	2.04372420
	L=2.0	
0	2.3703703	2.00823050
30	2.4043864	.84895980
60	2.3850705	1.21969930
90	2.2731527	2.11040950
120	2.1535473	1.00722740
150	2.0746584	1.04675550
180	2.0480000	2.06250670
	L=1.5	
0	3.0000000	1.87499980
30	2.8523364	.70436854
60	2.6151603	1.29023730
90	2.3760000	2.09808020
120	2.1970871	-.97324650
150	2.0950409	1.07984970
180	2.0625000	2.09374990
	L=1	
0	8.0000000	2.66666660
30	3.7352724	.46478701
60	2.6666662	1.33333440
90	2.3680000	2.03776010
120	2.1924199	-.90874990
150	2.1017732	1.131359
180	2.0740740	2.13991770
180	0.0000000	-.00000176

TABLE 2 : Stresses along the inhomogeneity boundary (Chapter XI), for a few values of radius a of the inhomogeneity and the distance l between the centres of the inclusion and the inhomogeneity : (Plane stress case, Poisson's ratio = 1/3).

NORMAL STRESS ($\frac{\sigma_n}{\mu_{m\delta}}$); $\beta = \infty$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=\delta=-\delta_2$	$\delta_1=\delta, \delta_2=0$	$\delta_1=0, \delta_2=\delta$
	0	-1.920	-4.333	-3.126	1.206
a = 1	30	.234	-2.073	-.919	1.153
1 = 2.25	60	.780	-.692	.044	.736
	90	.331	-.601	-.134	.466
	120	-.031	-.619	-.325	.293
	150	-.224	-.626	-.425	.200
	180	-.284	-.627	-.455	.171
		-.750	-1.951	-1.350	.600
a = 1	30	-.039	-1.083	-.561	.522
1 = 3	60	.397	-.459	-.030	.428
	90	.239	-.401	-.080	.320
	120	.008	-.424	-.207	.216
	150	-.138	-.436	-.287	.148
	180	-.187	-.439	-.313	.126
		-.333	-.900	-.616	.283
	30	-.061	-.579	-.320	.259
a = 1	60	.195	-.278	-.041	.236
1 = 4	90	.155	-.241	-.042	.198
	120	.020	-.267	-.123	.144
	150	-.083	-.286	-.185	.101
	180	-.120	-.292	-.206	.086
		-.120	-.327	-.223	.103
a = 1	30	-.036	-.239	-.138	.101
1 = 6	60	.071	-.131	-.029	.101
	90	.076	-.110	-.016	.093
	120	.017	-.129	-.055	.073
	150	-.039	-.148	-.093	.054
	180	-.061	-.154	-.107	.046
		-1.333	-3.157	-2.245	.912
a = 2	30	-.245	-2.357	-1.301	1.055
1 = 4	60	.781	-1.324	-.271	1.053
	90	.622	-1.033	-.205	.828
	120	.081	-1.051	-.485	.566
	150	-.333	-1.096	-.715	.381
	180	-.480	-1.113	-.796	.316
		-.750	-1.999	-1.374	.624
a = .5	30	.147	-.482	-.167	.315
1 = 2	60	.250	-.156	.046	.203
	90	.089	-.173	-.041	.131
	120	-.015	-.182	-.099	.083
	150	-.067	-.184	-.126	.058
	180	[illegible]	[illegible]	[illegible]	[illegible]

NORMAL STRESS $\left(\frac{\sigma_z}{\mu_m \delta}\right)$; $\beta = 3$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$	$\delta_1=\delta,\ \delta_2=0$	$\delta_1=0,\ \delta_2=\delta$
		-1.645	-2.828	-2.237	.591
a = 1	30	.200	-1.330	-.564	.765
l = 2.25	60	.668	-.449	.109	.559
	90	.284	-.446	-.081	.365
	120	-.026	-.480	-.253	.226
	150	-.192	-.491	-.341	.149
	180	-.243	-.493	-.368	.125
		-.642	-1.344	-.993	.350
a = 1	30	-.033	-.698	-.366	.332
l = 3	60	.341	-.277	.031	.309
	90	.205	-.277	-.035	.241
	120	.007	-.315	-.153	.161
	150	-.119	-.333	-.226	.107
	180	-.160	-.337	-.249	0.088
		-.285	-.632	-.459	.173
a = 1	30	-.052	-.383	-.217	.165
l = 4	60	.167	-.165	0.0	.166
	90	.133	-.158	-.012	.146
	120	.017	-.193	-.087	.105
	150	-.071	-.214	-.143	.071
	180	-.102	-.221	-.162	.059
		-.102	-.233	-.168	.065
a = 1	30	-.031	-.163	-.097	.065
l = 6	60	.061	-.079	-.008	.070
	90	.065	-.069	-.001	.067
	120	.015	-.090	-.037	.053
	150	-.035	-.109	-.071	.037
	180	-.052	-.115	-.083	.031
		-.642	-1.360	-1.001	.358
a = 2	30	.126	-.276	-.074	.201
l = 4	60	.214	-.101	.056	.158
	90	.077	-.134	-.028	.105
	120	-.015	-.145	-.079	.066
	150	-.058	-.147	-.102	.044
	180	-.071	-. 48	-.109	.038
		-1.142	-2.149	-1.646	.503
a = .5	30	-.210	-1.566	-.888	.678
l = 2	60	.669	-.844	-.087	.756
	90	.533	-.694	-.080	.614
	120	.069	-.757	-.343	.413
	150	-.286	-.816	-.551	.265
	180	-.411	-.835	-.623	.212

NORMAL STRESS ($\frac{\sigma_r}{k_w \delta}$); $\beta = 1/3$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta = -\delta_2$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
a = 1	0	-.768	-.847	-.807	.039
l = 2.25	30	.095	-.382	-.144	.238
	60	.312	-.133	.089	.222
	90	.132	-.171	-.019	.152
	120	-.012	-.197	-.104	.092
	150	-.089	-.205	-.147	.057
	180	-.115	-.206	-.160	.046
		-.300	-.452	-.376	.076
a = 1	30	-.015	-.202	-.109	.093
l = 3	60	.159	-.067	.045	.113
	90	.095	-.094	0.0	.095
	120	.003	-.121	-.059	.062
	150	-.055	-.135	-.094	.039
	180	-.075	-.136	-.105	.030
		-.133	-.220	-.177	.043
a = 1	30	-.024	-.118	-.071	.046
l = 4	60	.076	-.038	.019	.058
	90	.062	-.048	.006	.055
	120	.008	-.071	-.031	.039
	150	-.055	-.083	-.058	.025
	180	-.046	-.087	-.067	.019
		-.046	-.065	-.065	.017
a = 1	30	-.014	-.055	-.033	.019
l = 6	60	.028	-.019	.004	.024
	90	.030	-.018	.005	.024
	120	.007	-.031	-.012	.019
	150	-.015	-.041	-.028	.012
	180	-.024	-.044	-.034	.010
		-.300	-.446	-.373	.073
a = 2	30	.059	-.055	.001	.057
l = 4	60	.100	-.050	.034	.065
	90	.035	-.055	-.009	.045
	120	-.006	-.062	-.034	.027
	150	-.027	-.065	-.045	.018
	180	-.035	-.065	-.048	.015
		-.555	-.705	-.619	.086
a = .5	30	-.096	-.489	-.295	.195
l = 2	60	.312	-.258	.036	.275
	90	.249	-.221	.015	.235
	120	.032	-.276	-.121	.154
	150	-.135	-.514	-.224	.090

NORMAL STRESS $\left(\frac{\sigma_x}{\mu_m \delta}\right)$; $\beta = 1$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta,\ \delta_2 = 0$	$\delta_1 = 0,\ \delta_2 =$
		-1.280	-1.740	-1.510	.230
a = 1	30	.156	-.803	-.323	.479
1 = 2.25	60	.520	-.275	.122	.397
	90	.221	-.311	-.045	.266
	120	-.020	-.347	-.184	.163
	150	-.149	-.358	-.254	.104
	180	-.189	-.360	-.275	.085
		-.500	-.874	-.687	.187
a = 1	30	-.026	-.423	-.224	.198
1 = 3	60	.265	-.155	.054	.210
	90	.159	-.181	-.010	.170
	120	.005	-.220	-.107	.113
	150	-.092	-.237	-.165	.072
	180	-.125	-.242	-.183	.058
		-.222	-.419	-.320	.098
a = 1	30	-.040	-.239	-.140	.099
1 = 4	60	.130	-.090	.019	.110
	90	.103	-.098	.002	.101
	120	.013	-.131	-.059	.072
	150	-.055	-.151	-.103	.047
	180	-.080	-.156	-.118	.038
		-.080	-.156	-.118	.038
a = 1	30	-.024	-.104	-.064	.040
1 = 6	60	.047	-.044	.001	.046
	90	.051	-.040	.005	.046
	120	.011	-.060	-.024	.035
	150	-.026	-.075	-.050	.024
	180	-.040	-.080	-.060	.019
		-.500	-.874	-.687	.187
a = 2	30	.098	-.143	-.022	.121
1 = 4	60	.166	-.062	.052	.114
	90	.059	-.097	-.018	.078
	120	-.010	-.107	-.058	.048
	150	-.045	-.109	-.077	.032
	180	-.055	-.109	-.082	.027
		-.888	-1.382	-1.135	.246
a = .5	30	-.163	-.983	-.573	.409
1 = 2	60	.520	-.505	.007	.513
	90	.415	-.440	-.012	.427
	120	.054	-.514	-.229	.284
	150	-.222	-.570	-.396	.174
	180	-.320	-.588	-.454	.134

TANGENTIAL STRESS ($\frac{\tau_{r\theta}}{\mu_m \delta}$); $\beta = \infty$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		0.000	0.000	0.0	.000
a = 1	30	1.365	-.110	.627	.738
l = 2.25	60	.100	-.434	-.167	.267
	90	-.367	-.369	-.368	.001
	120	-.359	-.220	-.289	-.069
	150	-.200	-.099	-.150	-.050
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.623	.171	.397	.225
l = 3	60	.159	-.151	.003	.155
	90	-.180	-.223	-.201	.021
	120	-.230	-.158	-.194	-.035
	150	-.140	-.076	-.108	-.031
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.291	.137	.214	.077
l = 4	60	.122	-.025	.048	.074
	90	-.083	-.116	-.099	.016
	120	-.141	-.103	-.122	-.018
	150	-.093	-.055	-.074	-.019
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.106	.063	.084	.021
l = 6	60	.064	.015	.040	.024
	90	-.026	-.040	-.033	.007
	120	-.067	-.050	-.059	-.008
	150	-.049	-.031	-.040	-.009
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 2	30	.465	-.016	.224	.241
l = 4	60	0.000	-.198	-.099	.099
	90	-.120	-.128	-.124	.004
	120	-.106	-.068	-.087	-.018
	150	-.057	-.029	-.043	-.013
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = .5	30	1.165	.195	.680	.485
l = 2	60	.491	-.118	.186	.304
	90	-.332	-.363	-.347	.015
	120	-.565	-.335	-.450	-.115

TANGENTIAL STRESS ($\frac{\tau_{r\theta}}{\mu_m \delta}$); $\beta = 3$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		0.000	0.000	0.0	.000
a = 1, l = 2.25	30	1.170	.056	.613	.556
	60	.086	-.265	-.089	.175
	90	-.314	-.253	-.284	-.030
	120	-.308	-.153	-.230	-.077
	150	-.171	-.069	-.120	-.051
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1, l = 3	30	.534	.194	.364	.169
	60	.136	-.084	.025	.110
	90	-.154	-.160	-.157	.003
	120	-.197	-.117	-.157	-.040
	150	-.120	-.057	-.088	-.031
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1, l = 4	30	.249	.133	.191	.058
	60	.105	-.003	.051	.054
	90	-.071	-.085	-.078	.007
	120	-.121	-.079	-.1	.020
	150	-.080	-.042	-.061	-.018
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1, l = 6	30	.091	.057	.074	.016
	60	.055	.018	.036	.018
	90	-.022	-.030	-.026	.003
	120	-.057	-.040	-.049	-.008
	150	-.042	-.025	-.033	-.008
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 2, l = 4	30	.398	.047	.223	.175
	60	0.000	-.132	-.066	.066
	90	-.102	-.089	-.096	-.006
	120	-.090	-.047	-.069	-.021
	150	-.049	-.020	-.034	-.014
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = .5, l = 2	30	.999	.231	.615	.384
	60	.421	-.027	.197	.224
	90	-.284	-.254	-.269	-.015
	120	-.484	-.250	-.367	-.116
	150	-.320	-.139	-.230	-.090
	180	0.000	0.000	0.0	.000

TANGENTIAL STRESS ($\frac{\tau_{r\theta}}{\mu_m \delta}$) ; $\beta = 1/3$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta = -\delta_2$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		0.000	0.000	0.0	.000
a = 1	30	.546	.107	.326	.219
l = 2.25	60	.040	-.066	-.013	.053
	90	-.146	-.084	-.115	-.031
	120	-.143	-.052	-.098	-.045
	150	-.080	-.023	-.051	-.028
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.249	.116	.182	.066
l = 3	60	.063	-.015	.024	.039
	90	-.072	-.058	-.065	-.006
	120	-.092	-.044	-.068	-.023
	150	-.056	-.022	-.039	-.016
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.116	.070	.093	.022
l = 4	60	.049	.008	.028	.020
	90	-.033	-.032	-.032	0.000
	120	-.056	-.032	-.044	-.012
	150	-.037	-.017	-.027	-.009
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.042	.028	.035	.006
	60	.025	.011	.018	.007
l = 6	90	-.010	-.011	-.011	0.000
	120	-.026	-.017	-.022	-.004
	150	-.019	-.010	-.015	-.004
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 2	30	.186	.055	.120	.065
l = 4	60	0.000	-.042	-.021	.021
	90	-.048	-.030	-.039	-.008
	120	-.042	-.016	-.029	-.013
	150	-.022	-.006	-.014	-.008
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = .5	30	.466	.141	.304	.162
l = 2	60	.196	.026	.111	.085
	90	-.132	-.088	-.110	-.022
	120	-.226	-.097	-.161	-.064
	150	-.149	-.055	-.102	-.046
	180	0.000	0.000	0.0	.000

TANGENTIAL STRESS ($\frac{\tau_{r\theta}}{\mu_m \delta}$) ; $\beta = 1$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		0.000	0.000	0.0	.000
a = 1	30	.910	.122	.516	.393
l =2.25	60	.067	-.150	-.041	.108
	90	-.244	-.164	-.204	-.040
	120	-.239	-.100	-.170	-.069
	150	-.133	-.045	-.089	-.044
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.415	.176	.295	.119
l = 3	60	.106	-.042	.031	.074
	90	-.120	-.108	-.114	-.005
	120	-.153	-.081	-.117	-.035
	150	-.093	-.040	-.066	-.026
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.194	.111	.153	.041
l = 4	60	.081	.006	.044	.037
	90	-.055	-.059	-.057	.002
	120	-.094	-.057	-.075	-.018
	150	-.062	-.031	-.046	-.015
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 1	30	.071	.046	.058	.012
l = 6	60	.043	.016	.030	.013
	90	-.017	-.021	-.019	.001
	120	-.044	-.029	-.037	-.007
	150	-.033	-.018	-.025	-.007
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = 2	30	.310	.068	.189	.120
l = 4	60	0.000	-.084	-.042	.042
	90	-.080	-.059	-.069	-.010
	120	-.070	-.031	-.050	-.019
	150	-.038	-.013	-.025	-.012
	180	0.000	0.000	0.0	.000
		0.000	0.000	0.0	.000
a = .5	30	.777	.212	.495	.282
l = 2	60	.327	.016	.172	.155
	90	-.221	-.168	-.195	-.026
	120	-.377	-.176	-.276	-.100
	150	-.249	-.099	-.174	-.075
	180	0.000	0.000	0.0	.000

HOOP STRESS INSIDE $\left(\frac{(\sigma_\theta)_{in}}{\mu_m \delta}\right)$; $\beta = \infty$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=0, \delta_2=\delta$
		1.920	.003	.961	.958
a = 1	30	-.234	.021	-.106	-.127
l = 2.25	60	-.780	.209	-.285	-.495
	90	-.331	.507	.087	-.419
	120	.031	.620	.325	-.294
	150	.224	.653	.439	-.214
	180	.284	.660	.472	-.188
		.750	.395	.572	.177
a = 1	30	.039	.036	.037	.001
l = 3	60	-.397	.005	-.196	-.201
	90	-.239	.205	-.017	-.222
	120	-.008	.324	.157	-.166
	150	.138	.371	.255	-.116
	180	.187	.384	.285	-.098
		.333	.261	.297	.035
a = 1	30	.061	.074	.067	-.006
l = 4	60	-.195	-.016	-.105	-.089
	90	-.155	.075	-.039	-.115
	120	-.020	.164	.071	-.092
	150	.083	.209	.146	-.063
	180	.120	.222	.171	-.051
		.120	.118	.119	0.000
a = 1	30	.036	.056	.046	-.009
l = 6	60	-.071	-.003	-.037	-.034
	90	-.076	.016	-.029	-.046
	120	-.017	.061	.021	-.039
	150	.039	.092	.065	-.026
	180	.061	.102	.081	-.020
		.750	.249	.499	.250
a = 2	30	-.147	-.145	-.146	0.000
l = 4	60	-.250	.072	-.088	-.161
	90	-.089	.183	.046	-.136
	120	.015	.208	.111	-.096
	150	.067	.212	.140	-.072
	180	.083	.212	.148	-.064
		1.333	.601	.967	.365
a = .5	30	.245	.337	.291	-.046
l = 2	60	-.781	.147	-.316	-.464
	90	-.622	.372	-.125	-.497
	120	-.081	.636	.277	-.359
	150	.333	.787	.560	-.226
	180	.480	.833	.656	-.176

HOOP STRESS INSIDE $\left(\frac{(\sigma_\theta)_{in}}{\mu_m \delta}\right)$; $\beta = 1/3$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2$	$\delta_1=\delta, \delta_2=0$	$\delta_1=0, \delta_2=\delta$
		.768	.025	.396	.371
a = 1	30	-.093	-.113	-.103	.009
l = 2.25	60	-.312	-.139	-.225	-.086
	90	-.132	-.044	-.088	-.043
	120	.012	-.005	.003	.009
	150	.089	.005	.047	.042
	180	.113	.008	.060	.052
		.300	.115	.207	.092
a = 1	30	.015	-.061	-.022	.038
l = 3	60	-.159	-.111	-.135	-.023
	90	-.095	-.047	-.071	-.024
	120	-.003	-.006	-.005	.001
	150	.055	.010	.032	.022
	180	.075	.014	.044	.030
		.133	.065	.099	.033
a = 1	30	.024	-.018	.003	.021
l = 4	60	-.078	-.067	-.073	-.005
	90	-.062	-.039	-.050	-.011
	120	-.008	-.008	-.008	0.000
	150	.033	.008	.020	.012
	180	.048	.013	.030	.017
		.048	.025	.036	.011
a = 1	30	.014	-.001	.006	.008
l = 6	60	-.028	-.028	-.028	0.000
	90	-.030	-.022	-.026	-.003
	120	-.007	-.006	-.006	0.000
	150	.015	.004	.010	.005
	180	.024	.008	.016	.007
		.300	.132	.216	.083
a = 2	30	-.059	-.098	-.078	.019
l = 4	60	-.100	-.045	-.072	-.027
	90	-.035	-.007	-.021	-.014
	120	.006	.001	.003	.002
	150	.027	.002	.014	.012
	180	.033	.003	.018	.015
		.533	.083	.308	.224
a = .5	30	.098	-.056	.020	.077
l = 2	60	-.312	-.186	-.249	-.062
	90	-.249	-.129	-.189	-.059
	120	-.032	-.039	-.036	.003
	150	.133	.013	.073	.060

HOOP STRESS INSIDE ($\frac{(\sigma_\theta)_{iw}}{\mu_m \delta}$); $\beta = 1$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		1.280	.034	.657	.622
a = 1	30	-.156	-.143	-.149	-.006
l = 2.25	60	-.520	-.149	-.334	-.185
	90	-.221	.016	-.102	-.118
	120	.020	.084	.052	-.031
	150	.149	.104	.127	.022
	180	.189	.108	.148	.040
		.500	.208	.354	.145
a = 1	30	.026	-.073	-.023	.050
l = 3	60	-.265	-.143	-.204	-.060
	90	-.159	-.031	-.095	-.064
	120	-.005	.039	.016	-.022
	150	.092	.068	.080	.012
	180	.125	.075	.1	.024
		.222	.123	.172	.049
a = 1	30	.040	-.012	.014	.026
l = 4	60	-.130	-.090	-.110	-.019
	90	-.103	-.039	-.071	-.031
	120	-.013	.013	0.0	.013
	150	.055	.041	.048	.006
	180	.080	.050	.065	.014
		.080	.050	.065	.014
a = 1	30	.024	.006	.015	.008
l = 6	60	-.047	-.037	-.042	-.005
	90	-.051	-.027	-.039	-.011
	120	-.011	0.000	-.005	-.006
	150	.026	.019	.022	.003
	180	.040	.026	.033	.007
		.500	.208	.354	.145
a = 2	30	-.098	-.148	-.123	.025
l = 4	60	-.166	-.048	-.107	-.059
	90	-.059	.017	-.021	-.038
	120	.010	.032	.021	-.011
	150	.045	.035	.040	.004
	180	.055	.035	.045	.010
		.888	.197	.543	.345
		.163	-.023	.070	.093
		-.520	-.220	-.370	-.150
		-.415	-.113	-.264	-.151
		-.054	.042	-.005	-.048
		.222	.134	.178	.044
		.320	.162	.241	.078

HOOP STRESS OUTSIDE ($\frac{(\sigma_\theta)_{in}}{\mu_m \delta}$); $\beta = \infty$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$	$\delta_1=+\delta,\ \delta_2=0$	$\delta_1=0,\ \delta_2=\delta$
		-.639	1.968	.664	-1.304
a = 1	30	.078	1.203	.640	-.562
l = 2.25	60	.260	.617	.438	-.178
	90	.110	.389	.249	-.139
	120	-.010	.319	.154	-.164
	150	-.074	.299	.112	-.187
	180	-.094	.295	.1	.195
		-.249	.682	.216	-.466
a = 1	30	-.013	.633	.310	-.323
l = 3	60	.132	.445	.289	-.156
	90	.080	.292	.186	-.106
	120	.002	.221	.112	-.109
	150	-.046	.193	.073	-.120
	180	-.062	.186	.062	-.124
		-.111	.292	.090	-.201
a = 1	30	-.020	.310	.144	-.165
l = 4	60	.065	.270	.167	-.102
	90	.051	.196	.124	-.072
	120	.006	.146	.076	-.069
	150	-.027	.122	.047	-.075
	180	-.040	.115	.037	-.077
		-.039	.104	.032	-.072
a = 1	30	-.012	.116	.052	-.064
l = 6	60	.023	.120	.071	-.048
	90	.025	.099	.062	-.037
	120	.005	.076	.041	-.035
	150	-.013	.061	.024	-.037
	180	-.020	.057	.018	-.038
		-.249	.666	.208	-.458
a = 2	30	.049	.424	.236	-.187
l = 4	60	.083	.170	.126	-.043
	90	.030	.102	.066	-.036
	120	-.005	.088	.041	-.046
	150	-.022	.086	.032	-.054
	180	-.027	.086	.029	-.057
		-.444	1.317	.436	-.881
a = .5	30	-.081	1.227	.572	-.654
l = 2	60	.260	1.010	.635	-.374
	90	.207	.762	.485	-.277
	120	.027	.592	.309	-.282
	150	-.111	.507	.197	-.309
	180	-.160	.482	.161	-.321

HOOP STRESS OUTSIDE ($\frac{(\sigma_\theta)_m}{\delta_1 \mu_m}$); $\beta = 3$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		.182	.745	.464	-.281
a = 1	30	-.022	.427	.202	-.224
1 =2.25	60	-.074	.225	.075	-.149
	90	-.031	.198	.083	-.115
	120	.002	.196	.099	-.096
	150	.021	.195	.108	-.087
	180	.027	.195	.111	-.084
		.071	.333	.202	-.131
a = 1	30	.003	.227	.115	-.112
1 = 3	60	-.037	.141	.051	-.089
	90	-.022	.124	.050	-.073
	120	0.000	.122	.060	-.061
	150	.013	.121	.067	-.054
	180	.017	.121	.069	-.051
		.031	.158	.094	-.063
a = 1	30	.005	.121	.063	-.057
1 = 4	60	-.018	.083	.032	-.050
	90	-.014	.073	.029	-.044
	120	-.001	.073	.035	-.037
	150	.007	.074	.041	-.033
	180	.011	.074	.042	-.031
		.011	.059	.035	-.024
a = 1	30	.003	.050	.026	-.023
1 = 6	60	-.006	.037	.015	-.022
	90	-.007	.034	.013	-.020
	120	-.001	.034	.016	-.018
	150	.003	.035	.019	-.016
	180	.005	.036	.021	-.015
		.071	.313	.192	-.120
a = 2	30	-.014	.117	.051	-.065
1 = 4	60	-.023	.062	.019	-.042
	90	-.008	.059	.025	-.034
	120	.001	.059	.030	-.029
	150	.006	.059	.033	-.026
	180	.007	.059	.033	-.025
		.127	.590	.358	-.231
a = .5	30	.023	.489	.256	-.233
1 = 2	60	-.074	.353	.139	-.213
	90	-.059	.302	.121	-.180
	120	-.007	.292	.142	-.150
	150	.031	.292	.162	-.130
	180	.045	.292	.169	-.123

HOOP STRESS OUTSIDE ($\frac{(\sigma_3)_m}{\mu_m \delta}$); $\beta = 1/3$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=\delta=-\delta_2$	$\delta_1=\delta, \delta_2=0$	$\delta_1=0, \delta_2=\delta$
		2.815	-.332	1.241	1.574
a = 1	30	-.343	-.626	-.484	.141
1=2.25	60	-1.144	-.570	-.857	-.286
	90	-.486	-.188	-.337	-.148
	120	.045	-.036	.004	.041
	150	.329	.008	.169	.160
	180	.416	.017	.217	.199
		1.099	.266	.683	.416
a = 1	30	.057	-.332	-.137	.195
1 = 3	60	-.583	-.460	-.521	-.061
	90	-.352	-.203	-.277	-.074
	120	-.013	-.042	-.027	.014
	150	.203	.022	.113	.090
	180	.274	.038	.156	.118
		.488	.173	.331	.157
a = 1	30	.089	-.118	-.014	.104
1 = 4	60	-.286	-.281	-.284	-.002
	90	-.228	-.164	-.196	-.031
	120	-.029	-.044	-.037	.007
	150	.122	.018	.070	.051
	180	.176	.037	.106	.069
		.175	.070	.123	.052
a = 1	30	.053	-.025	.014	.039
1 = 6	60	-.105	-.119	-.112	.006
	90	-.112	-.095	-.103	-.008
	120	-.026	-.033	-.029	.003
	150	.057	.009	.033	.023
	180	.089	.024	.057	.032
		1.099	.313	.706	.393
a = 2	30	-.216	-.424	-.320	.104
1 = 4	60	-.366	-.180	-.273	-.092
	90	-.132	-.031	-.081	-.050
	120	.022	.002	.012	.009
	150	.099	.008	.053	.045
	180	.122	.009	.065	.056
		1.955	.044	.999	.955
a = .5	30	.359	-.418	-.029	.389
1 = 2	60	-1.145	-.815	-.980	-.164
	90	-.913	-.558	-.735	-.177
	120	-.119	-.205	-.162	.042
	150	.489	.001	.245	.244
	180	.704	.063	.383	.320

HOOP STRESS OUTSIDE ($\frac{(\sigma_s)_m}{\mu_m \delta}$); $\beta = \phi\ 1$

	θ	$\delta_1 = \delta_2 = \delta$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = 0, \delta_2 = \delta$
		1.280	.034	.657	.622
a = 1	30	-.156	-.143	-.149	-.006
l = 2.25	60	-.520	-.149	-.334	-.185
	90	-.221	.016	-.102	-.118
	120	.020	.084	.052	-.031
	150	.149	.104	.127	.022
	180	.189	.108	.148	.040
		.500	.208	.354	.145
a = 1	30	.026	-.073	-.023	.050
l = 3	60	-.265	-.143	-.204	-.060
	90	-.159	-.031	-.095	-.064
	120	-.005	.039	.016	-.022
	150	.092	.068	.080	.012
	180	.125	.075	.1	.024
		.222	.123	.172	.049
a = 1	30	.040	-.012	.014	.026
l = 4	60	-.130	-.090	-.110	-.019
	90	-.103	-.039	-.071	-.031
	120	-.013	.013	0.0	.013
	150	.055	.041	.048	.006
	180	.080	.050	.065	.014
		.080	.050	.065	.014
a = 1	30	.024	.006	.015	.008
l = 6	60	-.047	-.037	-.042	-.005
	90	-.051	-.027	-.039	-.011
	120	-.011	0.000	-.005	-.006
	150	.026	.019	.022	.003
	180	.040	.026	.033	.007
		.500	.208	.354	.145
a = 2	30	-.098	-.148	-.123	.025
l = 4	60	-.166	-.048	-.107	-.059
	90	-.059	.017	-.021	-.038
	120	.010	.032	.021	-.011
	150	.045	.035	.040	.004
	180	.055	.035	.045	.010
		.888	.197	.543	.345
a = .5	30	.163	-.023	.070	.093
l = 2	60	-.520	-.220	-.370	-.150
	90	-.415	-.113	-.264	-.151
	120	-.054	.042	-.005	-.048
	150	.222	.134	.178	.044
	180	.320	.162	.241	.078

HOOP STRESS OUTSIDE $\left(\frac{(\sigma_\theta)_m}{\delta E t_m}\right)$; $\beta = 0$

	θ	$\delta_1=\delta_2=\delta$	$\delta_1=-\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=0, \delta_2=\delta$
		5.119	-.390	2.364	2.755
a = 1	30	-.624	-1.101	-.863	.238
l = 2.25	60	-2.081	-1.123	-1.602	-.478
	90	-.884	-.459	-.671	-.212
	120	.083	-.189	-.053	.136
	150	.599	-.109	.244	.354
	180	.757	-.093	.331	.425
		1.999	.537	1.268	.731
a = 1	30	.105	-.591	-.243	.348
l = 3	60	-1.061	-.870	-.965	-.095
	90	-.640	-.423	-.531	-.108
	120	-.023	-.139	-.081	.057
	150	.370	-.023	.173	.197
	180	.499	.005	.252	.247
		.888	.327	.608	.280
a = 1	30	.163	-.215	-.026	.189
l = 4	60	-.520	-.528	-.524	.003
	90	-.415	-.326	-.370	-.044
	120	-.054	-.111	-.082	.028
	150	.222	.001	.111	.110
	180	.320	.033	.176	.143
		.319	.126	.223	.096
a = 1	30	.097	-.048	.024	.072
l = 6	60	-.191	-.223	-.207	.015
	90	-.204	-.183	-.193	-.010
	120	-.047	-.073	-.060	.012
	150	.104	.005	.054	.049
	180	.163	.031	.097	.065
		1.999	.666	1.333	.666
a = 2	30	-.393	-.762	-.577	.184
l = 4	60	-.666	-.361	-.513	-.152
	90	-.240	-.097	-.168	-.071
	120	.040	-.036	.002	.038
	150	.180	-.025	.077	.103
	180	.222	-.024	.098	.123
		3.555	.123	1.839	1.716
a = .5	30	.653	-.760	-.053	.706
l = 2	60	-2.082	-1.547	-1.815	-.267
	90	-1.660	-1.119	-1.389	-.270
	120	-.217	-.496	-.357	.139
	150	.890	-.129	.380	.510
	180	1.280	-.018	.630	.649

TABLE 3 : Stresses along the inclusion boundary in the presence of an inhomogeneity (Chapter XI), for a few values of radius a of the inhomogeneity and the distance 1 between the centres of the inclusion and the inhomogeneity : (Plane stress case ; Poisson's ratio = 1/3).

NORMAL STRESS $\left(\frac{(\sigma_{rr})}{\mu_m \delta}\right)$; $\beta=\infty$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
	0	-1.799	-1.150	-.501	-.646
a = 1	30	-1.799	-.994	-.189	-.805
l = 2.25	60	-1.805	-.690	.425	-1.115
	90	-1.839	-.576	.685	-1.262
	120	-1.959	-.862	.245	-1.107
	150	-1.962	-1.172	-.382	-.789
	180	-.491	-.247	-.003	-.244
		-1.934	-1.245	-.556	-.688
a = 1	30.	-1.936	-1.085	-.234	-.850
l = 3	60	-1.944	-.770	.404	-1.174
	90	-1.967	-.634	.697	-1.332
	120	-2.007	-.840	.327	-1.167
	150	-1.943	-1.109	-.276	-.833
	180	-1.716	-1.071	-.426	-.644
		-1.977	-1.296	-.615	-.681
a = 1	30	-1.979	-1.132	-.286	-.846
l = 4	60	-1.984	-.807	.369	-1.176
	90	-1.993	-.652	.688	-1.341
	120	-2.001	-.926	.348	-1.175
	150	-1.976	-1.134	-.291	-.842
	180	-1.936	-1.262	-.588	-.674
		-1.995	-1.323	-.651	-.671
a = 1	30	-1.995	-1.157	-.319	-.838
l = 6	60	-1.997	-.826	.345	-1.171
	90	-1.999	-.661	.675	-1.337
	120	-1.999	-.822	.342	-1.171
	150	-1.995	-1.157	-.319	-.837
	180	-1.990	-1.319	-.648	-.671
		-1.876	-1.037	-.197	-.839
a = 2	30	-1.882	-.892	.097	-.989
l = 4	60	-1.904	-.619	.665	-1.284
	90	-1.950	-.534	.881	-1.415
	120	-1.995	-.755	.486	-1.242
	150	-1.861	-.933	-.006	-.927
	180	-1.597	-.822	-.047	-.774
		-1.939	-1.279	-.620	-.659
a = .5	30	-1.939	-1.116	-.293	-.823
l = 2	60	-1.940	-.790	.360	-1.150
	90	-1.949	-.637	.675	-1.312
	120	-1.994	-.844	.305	-1.149
	150	-2.039	-1.204	-.369	-.834
	180	-1.434	[illegible]	[illegible]	[illegible]

NORMAL STRESS ($\frac{\sigma_n}{\mu_m \delta}$); $\beta = 3$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		-2.100	-1.428	-.755	-.672
a = 1	30	-2.100	-1.255	-.411	-.844
l=2.25	60	-2.097	-.907	.282	-1.169
	90	-2.0[illegible]0	-.710	.659	-1.369
	120	-2.015	-.804	.405	-1.210
	150	-2.018	-1.133	-.247	-.885
	180	-2.754	-1.965	-1.176	-.708
		-2.032	-1.377	-.722	-.655
a = 1	30	-2.031	-1.208	-.384	-.823
l = 3	60	-2.027	-.865	.297	-1.162
	90	-2.016	-.681	.652	-1.334
	120	-1.996	-.827	.341	-1.168
	150	-2.028	-1.194	-.359	-.834
	180	-2.142	-1.470	-.798	-.671
		-2.011	-1.351	-.692	-.659
a = 1	30	-2.010	-1.183	-.356	-.826
l = 4	60	-2.007	-.846	.315	-1.161
	90	-2.003	-.673	.656	-1.329
	120	-1.999	-.836	.326	-1.162
	150	-2.011	-1.182	-.354	-.828
	180	-2.031	-1.369	-.707	-.662
		-2.002	-1.338	-.674	-.664
a = 1	30	-2.002	-1.171	-.340	-.830
l = 6	60	-2.001	-.836	.327	-1.164
	90	-2.000	-.669	.662	-1.331
	120	-2.000	-.835	.328	-1.164
	150	-2.002	-1.171	-.340	-.831
	180	-2.004	-1.340	-.676	-.664
		-2.061	-1.484	-.906	-.577
a = 2	30	-2.056	-1.305	-.553	-.752
l = 4	60	-2.047	-.940	.166	-1.107
	90	-2.024	-.729	.566	-1.295
	120	-2.000	-.763	.273	-1.137
	150	-2.069	-1.282	-.494	-.787
	180	-2.201	-1.606	-1.011	-.595
		-2.030	-1.360	-.690	-.669
a =.5	30	-2.030	-1.192	-.353	-.839
l = 2	60	-2.029	-.855	.319	-1.174
	90	-2.025	-.681	.662	-1.343
	120	-2.003	-.836	.350	-1.176
	150	-1.9[illegible]0	-1.144	-.308	-.836
	180	-2.[illegible]2	-1.560	-.837	-.722

NORMAL STRESS ($\frac{\sigma_n}{\mu_m \delta}$) ; $\beta = 1/3$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta;\ \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0,\ \delta_2 = \delta$
		-2.057	-1.386	-.716	-.670
a=1	30	-2.057	-1.216	-.378	-.840
1=2.25	60	-2.055	-.874	.305	-1.160
	90	-2.045	-.691	.662	-1.353
	120	-2.008	-.820	.367	-1.180
	150	-2.010	-1.154	-.298	-.856
	180	-2.430	-1.674	-.917	-.756
		-2.018	-1.350	-.695	-.660
a = 1	30	-2.016	-1.190	-.362	-.828
1 = 3	60	-2.015	-.851	.312	-1.164
	90	-2.009	-.675	.659	-1.333
	120	-1.997	-.830	.336	-1.167
	150	-2.016	-1.182	-.348	-.833
	180	-2.031	-1.410	-.739	-.670
		-2.006	-1.343	-.681	-.662
a = 1	30	-2.005	-1.176	-.346	-.829
1 = 4	60	-2.004	-.840	.322	-1.163
	90	-2.001	-.670	.660	-1.331
	120	-1.999	-.835	.329	-1.154
	150	-2.006	-1.175	-.345	-.830
	180	-2.018	-1.353	-.689	-.664
		-2.001	-1.336	-.671	-.665
a = 1	30	-2.001	-1.169	-.337	-.831
1 = 6	60	-2.000	-.835	.329	-1.165
	90	-2.000	-.668	.664	-1.332
	120	-2.000	-.834	.330	-1.165
	150	-2.001	-1.169	-.337	-.832
	180	-2.002	-1.337	-.671	-.665
		-2.035	-1.418	-.802	-.616
a =2	30	-2.033	-1.245	-.457	-.737
1 =4	60	-2.027	-.894	.23	-1.132
	90	-2.014	-.703	.607	-1.310
	120	-2.000	-.852	.295	-1.147
	150	-2.039	-1.232	-.426	-.806
	180	-2.115	-1.485	-.755	-.629
		-2.017	-1.348	-.680	-.668
a=.5	30	-2.017	-1.181	-.344	-.836
1= 2	60	-2.016	-.845	.325	-1.171
	90	-2.014	-.674	.664	-1.339
	120	-2.001	-.829	.342	-1.171
	150	-1.9[illegible]	-1.154	-.320	-.834
	180	-2.141	-1.460	-.759	-.700

NORMAL STRESS ($\frac{\sigma_m}{\mu_m \delta}$); $\beta = 1$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		-1.919	-1.259	-.599	-.659
a = 1	30	-1.919	-1.097	-.274	-.822
1 =2.25	60	-1.922	-.775	.370	-1.146
	90	-1.935	-.630	.674	-1.305
	120	-1.957	-.846	.294	-1.141
	150	-1.9[illegible]4	-1.172	-.360	-.812
	180	-1.396	-.880	-.380	-.507
		-1.973	-1.298	-.622	-.675
a = 1	30	-1.974	-1.133	-.293	-.840
1 = 3	60	-1.977	-.808	.361	-1.169
	90	-1.987	-.654	.676	-1.332
	120	-2.003	-.834	.330	-1.166
	150	-1.977	-1.144	-.310	-.833
	180	-1.996	-1.227	-.569	-.650
		-1.991	-1.318	-.646	-.672
a = 1	30	-1.991	-1.153	-.314	-.838
1 = 4	60	-1.993	-.822	.347	-1.170
	90	-1.997	-.661	.675	-1.336
	120	-2.000	-.830	.339	-1.169
	150	-1.990	-1.153	-.316	-.836
	180	-1.974	-1.304	-.635	-.669
		-1.998	-1.329	-.660	-.668
a = 1	30	-1.997	-1.162	-.327	-.835
1 = 6	60	-1.998	-.830	.336	-1.168
	90	-1.999	-.664	.670	-1.335
	120	-1.999	-.831	.336	-1.166
	150	-1.998	-1.162	-.327	-.835
	180	-1.996	-1.327	-.659	-.668
		-1.950	-1.214	-.478	-.736
a = 2	30	-1.953	-1.056	-.160	-.896
1 = 4	60	-1.961	-.747	.466	-1.213
	90	-1.9[illegible]0	-.614	.751	-1.365
	120	-1.999	-.803	.392	-1.195
	150	-1.944	-1.073	-.202	-.870
	180	-1.[illegible]3	-1.126	-.414	-.711
		-1.975	-1.311	-.648	-.663
a = .5	30	-1.975	-1.146	-.317	-.829
1 = 2	60	-1.978	-.81[illegible]	.344	-1.160
	90	-1.979	-.654	.670	-1.325
	120	-1.997	-.837	.321	-1.159
	150	-2.015	-1.1[illegible]	-.34[illegible]	-.833
	180	-1.773	-1.15[illegible]	-.543	-.615

TANGENTIAL STRESS ($\frac{q_{ms}}{\mu_m \delta}$) ; $\beta = \infty$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
		0.000	0.000	0.000	0.000
a = 1	30	-.034	.256	.547	-.291
l =2.25	60	-.0[illegible]2	.217	.516	-.299
	90	-.157	-.123	-.0[illegible]9	-.034
	120	-.[illegible]35	-.444	-.[illegible]53	.20[illegible]
	150	.077	-.151	-.3[illegible]0	.229
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.01[illegible]	.272	.561	-.2[illegible]9
l = 3	60	-.034	.25[illegible]	.547	-.291
	90	-.051	-.043	-.03[illegible]	-.007
	120	-.030	-.30[illegible]	-.5[illegible]7	.27[illegible]
	150	.090	-.201	-.493	.292
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.00[illegible]	.2[illegible]2	.570	-.2[illegible]
l = 4	60	-.012	.276	.565	-.2[illegible]9
	90	-.013	-.012	-.010	-.001
	120	0.000	-.286	-.573	.2[illegible]7
	150	.02[illegible]	-.262	-.552	.2[illegible]9
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.001	.2[illegible]7	.575	-.2[illegible]
l = 6	60	-.002	.2[illegible]	.574	-.2[illegible]
	90	-.001	-.001	-.001	0.000
	120	.001	-.2[illegible]7	-.575	.2[illegible]8
	150	.004	-.2[illegible]4	-.573	.2[illegible]8
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 2	30	-.033	.257	.54[illegible]	-.291
l = 4	60	-.0[illegible]	.229	.524	-.295
	90	-.0[illegible]2	-.070	-.059	-.011
	120	-.0[illegible]3	-.302	-.5[illegible]1	.27[illegible]
	150	.132	-.15[illegible]	-.449	.290
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a =.5	30	-.010	.27[illegible]	.567	-.28[illegible]
l = 2	60	-.0[illegible]5	.2[illegible]5	.555	-.290
	90	-.050	-.041	-.032	-.00[illegible]
	120	-.0[illegible]1	-.344	-.60[illegible]	.263
	150	.04[illegible]	-.2[illegible]	-.504	.276

TANGENTIAL STRESS ($\frac{\tau_{rrs}}{\mu_m \delta}$); $\beta = 3$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta,\ \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0,\ \delta_2=\delta$
		0.000	0.000	0.000	0.000
a = 1	30	-.034	.2[illegible]0	.595	-.315
l = 2.25	60	-.0[illegible]	.[illegible]67	.61[illegible]	-.349
	90	-.157	-.04[illegible]	.059	-.10[illegible]
	120	-.235	-.379	-.522	.143
	150	.077	-.332	-.742	.410
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.01[illegible]	.2[illegible]4	.5[illegible]5	-.301
l = 3	60	-.034	.279	.593	-.314
	90	-.051	-.015	.020	-.035
	120	-.030	-.301	-.572	.270
	150	.090	-.26[illegible]	-.62[illegible]	.359
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.00[illegible]	.287	.580	-.293
l = 4	60	-.012	.285	.583	-.29[illegible]
	90	-.013	-.004	.005	-.009
	120	0.000	-.289	-.579	.289
	150	.026	-.282	-.590	.30[illegible]
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.001	.288	.57[illegible]	-.289
l = 6	60	-.002	.287	.57[illegible]	-.290
	90	-.001	0.000	0.000	-.001
	120	.001	-.288	-.57[illegible]	.289
	150	.004	-.287	-.579	.291
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 2	30	-.033	.2[illegible]1	.595	-.314
l = 4	60	-.0[illegible]	.272	.611	-.33[illegible]
	90	-.0[illegible]	-.022	.037	-.060
	120	-.0[illegible]3	-.301	-.57[illegible]	.277
	150	.13[illegible]	-.26[illegible]	-.669	.400
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = .5	30	-.010	.2[illegible]	.5[illegible]4	-.29[illegible]
l = 2	60	-.025	.2[illegible]1	.5[illegible]	-.307
	90	-.050	-.01[illegible]	.017	-.034
	120	-.0[illegible]1	-.320	-.559	.239
	150	.04[illegible]	-.2[illegible]	-.[illegible]20	.334
	180	0.000	0.000	0.000	0.000

TANGENTIAL STRESS $\left(\cdot \frac{q_{ms}}{\mu_m \delta}\right)$; $\beta = 1/3$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
a= 1		0.000	0.000	0.000	0.000
l=2.25	30	-.034	.278	.587	-.310
	60	-.0[illegible]2	.257	.597	-.339
	90	-.157	-.063	.030	-.094
	120	-.235	-.392	-.550	.157
	150	.077	-.289	-.656	.367
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.016	.282	.582	-.299
l = 3	60	-.034	.275	.586	-.310
	90	-.051	-.020	.011	-.031
	120	-.030	-.302	-.574	.271
	150	.090	-.256	-.604	.347
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a = 1	30	-.008	.286	.579	-.297
l = 4	60	-.012	.284	.590	-.296
	90	-.013	-.005	.003	-.003
	120	0.000	-.288	-.570	.289
	150	.026	-.279	-.584	.305
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a= 1	30	-.001	.288	.577	-.289
l= 6	60	-.002	.287	.578	-.290
	90	-.001	0.000	0.000	-.001
	120	.001	-.288	-.577	.289
	150	.004	-.287	-.578	.291
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a =2	30	-.033	.276	.586	-.310
l= 4	60	-.066	.264	.595	-.330
	90	-.082	-.031	.019	-.051
	120	-.023	-.300	-.577	.276
	150	.132	-.245	-.623	.377
	180	0.000	0.000	0.000	0.000
		0.000	0.000	0.000	0.000
a=.5	30	-.010	.284	.580	-.295
l= 2	60	-.025	.279	.593	-.304
	90	-.050	-.020	.009	-.030
	120	-.0[illegible]1	-.324	-.567	.243
	150	.04[illegible]	-.27[illegible]	-.60[illegible]	.324
	180	0.000	0.000	0.000	0.000

TANGENTIAL STRESS ($\frac{\gamma m s}{\mu_m \delta}$); $\beta = 1$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = +\delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		.000	0.000	0.000	0.0
a = 1	30	-.034	.265	.564	-.299
l = 2.25	60	-.082	.234	.551	-.317
	90	-.157	-.097	-.037	-.060
	120	-.235	-.422	-.609	.186
	150	.077	-.206	-.491	.284
	180	0.000	0.000	0.000	0.0
		.000	0.000	0.000	0.0
a = 1	30	-.016	.277	.571	-.293
l = 3	60	-.034	.265	.565	-.3
	90	-.051	-.033	-.014	-.018
	120	-.030	-.306	-.581	.275
	150	.090	-.225	-.542	.316
	180	0.000	0.000	0.000	0.0
		.000	0.000	0.000	0.0
a = 1	30	-.006	.284	.574	-.290
l = 4	60	-.012	.280	.572	-.292
	90	-.013	-.009	-.004	-.004
	120	0.000	-.287	-.575	.288
	150	.026	-.270	-.567	.296
	180	0.000	0.000	0.000	0.0
		.000	0.000	0.000	0.0
a = 1	30	-.001	.287	.576	-.289
l = 6	60	-.002	.286	.576	-.289
	90	-.001	-.001	0.000	0.0
	120	.001	-.287	-.576	.289
	150	.004	-.285	-.575	.290
	180	0.000	0.000	0.000	0.0
		.000	0.000	0.000	0.0
a = 2	30	-.033	.266	.565	-.299
l = 4	60	-.066	.244	.555	-.310
	90	-.082	-.053	-.024	-.029
	120	-.023	-.301	-.578	.277
	150	.132	-.195	-.522	.327
	180	0.000	0.000	0.000	0.0
		.000	0.000	0.000	0.0
a = .5	30	-.010	.281	.573	-.291
l = 2	60	-.025	.271	.568	-.297
	90	-.050	-.031	-.013	-.018
	120	-.081	-.335	-.589	.254
	150	.048	-.249	-.547	.297
	180	0.000	0.000	0.000	0.0

HOOP STRESS INSIDE ($\frac{(\sigma_3)_i}{\mu_m \delta}$) ; $\beta = \infty$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
		-2.000	-.642	.714	-1.357
a = 1	30	-2.000	-.815	.370	-1.185
l = 2.25	60	-1.997	-1.153	-.309	-.843
	90	-1.963	-1.294	-.624	-.669
	120	-1.780	-.984	-.189	-.795
	150	-1.258	-.317	.622	-.940
	180	-1.072	-.227	.617	-.845
		-1.999	-.637	.723	-1.361
a = 1	30	-1.996	-.804	.387	-1.191
l = 3	60	-1.984	-1.133	-.283	-.850
	90	-1.949	-1.276	-.603	-.673
	120	-1.874	-1.050	-.226	-.823
	150	-1.838	-.694	.450	-1.144
	180	-1.964	-.642	.679	-1.321
		-1.999	-.651	.697	-1.348
a = 1	30	-1.997	-.816	.364	-1.181
l = 4	60	-1.991	-1.145	-.300	-.845
	90	-1.977	-1.301	-.626	-.675
	120	-1.959	-1.120	-.281	-.839
	150	-1.968	-.795	.377	-1.172
	180	-1.997	-.654	.687	-1.342
		-1.999	-.661	.677	-1.338
a = 1	30	-1.999	-.827	.344	-1.171
l = 6	60	-1.997	-1.159	-.321	-.838
	90	-1.994	-1.323	-.652	-.671
	120	-1.992	-1.155	-.318	-.837
	150	-1.996	-.825	.345	-1.171
	180	-1.999	-.661	.676	-1.338
		-1.998	-.478	1.041	-1.519
a = 2	30	-1.990	-.652	.684	-1.337
l = 4	60	-1.959	-.995	-.030	-.964
	90	-1.889	-1.137	-.384	-.752
	120	-1.777	-.886	.004	-.891
	150	-1.765	-.523	.719	-1.242
	180	-1.902	-.465	.972	-1.437
		-2.000	-.664	.671	-1.335
a = .5	30	-2.000	-.832	.335	-1.167
l = 2	60	-1.999	-1.165	-.331	-.833
	90	-1.988	-1.320	-.652	-.668
	120	-1.924	-1.100	-.277	-.823
	150	-1.722	-.632	.456	-1.089
	180	-1.912	-.656	.599	-1.256

HOOP STRESS INSIDE $\left(\frac{(\sigma_3)_i}{\mu_m \delta}\right)$; $\beta = 3$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta,\ \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0,\ \delta_2=\delta$
		-1.999	-.675	.649	-1.324
a = 1	30	-1.999	-.839	.320	-1.160
l = 2.25	60	-2.001	-1.170	-.340	-.830
	90	-2.018	-1.354	-.690	-.663
	120	-2.109	-1.271	-.433	-.837
	150	-2.370	-1.121	.127	-1.249
	180	-2.463	-.797	.869	-1.666
		-2.000	-.680	.639	-1.319
a = 1	30	-2.001	-.846	.307	-1.154
l = 3	60	-2.007	-1.182	-.357	-.824
	90	-2.025	-1.362	-.699	-.662
	120	-2.062	-1.227	-.391	-.835
	150	-2.080	-.903	.272	-1.176
	180	-2.018	-.672	.672	-1.345
		-2.000	-.674	.651	-1.325
a = 1	30	-2.001	-.841	.318	-1.159
l = 4	60	-2.004	-1.177	-.349	-.827
	90	-2.011	-1.349	-.687	-.662
	120	-2.020	-1.190	-.359	-.830
	150	-2.015	-.852	.311	-1.163
	180	-2.001	-.672	.657	-1.329
		-2.000	-.669	.661	-1.330
a = 1	30	-2.000	-.836	.327	-1.164
l = 6	60	-2.001	-1.170	-.339	-.830
	90	-2.002	-1.338	-.673	-.664
	120	-2.003	-1.172	-.341	-.831
	150	-2.001	-.837	.327	-1.164
	180	-2.000	-.669	.661	-1.331
		-2.000	-.758	.484	-1.242
a = 2	30	-2.004	-.921	.162	-1.083
l = 4	60	-2.020	-1.251	-.483	-.768
	90	-2.055	-1.434	-.814	-.620
	120	-2.111	-1.315	-.519	-.795
	150	-2.117	-.989	.138	-1.127
	180	-2.048	-.750	.548	-1.298
		-1.999	-.667	.664	-1.332
a = .5	30	-1.999	-.833	.332	-1.166
l = 2	60	-2.000	-1.167	-.333	-.833
	90	-2.005	-1.339	-.674	-.665
	120	-2.037	-1.201	-.364	-.836
	150	-2.138	-.936	.265	-1.202
	180	-2.043	-.662	.719	-1.381

HOOP STRESS INSIDE $\left(\frac{(\sigma_3)_i}{\mu_m \delta}\right)$; $\beta = 1/3$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		-1.999	-.672	.655	-1.327
a = 1	30	-1.999	-.837	.324	-1.162
l = 2.25	60	-2.000	-1.169	-.338	-.831
	90	-2.010	-1.345	-.679	-.665
	120	-2.062	-1.223	-.384	-.839
	150	-2.211	-.991	.229	-1.220
	180	-2.264	-.761	.741	-1.503
		-2.000	-.674	.650	-1.325
a = 1	30	-2.000	-.841	.318	-1.159
l = 3	60	-2.004	-1.175	-.347	-.828
	90	-2.014	-1.349	-.685	-.664
	120	-2.035	-1.200	-.365	-.835
	150	-2.046	-.873	.299	-1.172
	180	-2.010	-.671	.667	-1.338
		-2.000	-.671	.657	-1.328
a = 1	30	-2.000	-.838	.324	-1.162
l = 4	60	-2.002	-1.172	-.342	-.829
	90	-2.006	-1.342	-.678	-.664
	120	-2.011	-1.179	-.348	-.831
	150	-2.009	-.844	.320	-1.164
	180	-2.000	-.669	.661	-1.330
		-2.000	-.668	.663	-1.331
a = 1	30	-2.000	-.834	.330	-1.165
l = 6	60	-2.000	-1.168	-.336	-.831
	90	-2.001	-1.336	-.670	-.665
	120	-2.002	-1.169	-.337	-.832
	150	-2.001	-.835	.329	-1.165
	180	-2.000	-.668	.663	-1.332
		-2.000	-.719	.561	-1.280
a = 2	30	-2.002	-.884	.234	-1.118
l = 4	60	-2.011	-1.215	-.419	-.795
	90	-2.031	-1.390	-.749	-.640
	120	-2.063	-1.249	-.435	-.813
	150	-2.066	-.922	.222	-1.144
	180	-2.027	-.718	.591	-1.309
		-1.999	-.667	.665	-1.332
a = .5	30	-1.999	-.833	.332	-1.166
l = 2	60	-2.000	-1.166	-.333	-.833
	90	-2.003	-1.337	-.670	-.666
	120	-2.021	-1.185	-.350	-.835
	150	-2.079	-.891	.295	-1.187
	180	-2.024	-.666	.692	-1.358

HOOP STRESS INSIDE ($\frac{(\sigma_s)_i}{\mu_m \delta}$); $\beta = 1$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		-2.000	-.657	.684	-1.342
a = 1	30	-2.000	-.826	.347	-1.173
l =2.25	60	-1.999	-1.161	-.324	-.837
	90	-1.985	-1.317	-.649	-.667
	120	-1.912	-1.092	-.272	-.819
	150	-1.703	-.623	.456	-1.079
	180	-1.629	-.501	.626	-1.127
		-1.999	-.655	.689	-1.344
a = 1	30	-1.998	-.822	.354	-1.176
l =3	60	-1.993	-1.153	-.313	-.840
	90	-1.979	-1.310	-.641	-.669
	120	-1.949	-1.119	-.289	-.829
	150	-1.935	-.777	.380	-1.157
	180	-1.985	-.657	.670	-1.327
		-1.999	-.660	.679	-1.339
a = 1	30	-1.999	-.826	.345	-1.172
l = 4	60	-1.996	-1.158	-.320	-.838
	90	-1.990	-1.320	-.650	-.670
	120	-1.983	-1.148	-.312	-.835
	150	-1.987	-.818	.350	-1.169
	180	-1.998	-.661	.674	-1.336
		-1.999	-.664	.670	-1.335
a = 1	30	-1.999	-.831	.337	-1.168
l = 6	60	-1.998	-1.163	-.328	-.835
	90	-1.997	-1.329	-.660	-.668
	120	-1.997	-1.162	-.327	-.834
	150	-1.998	-.830	.338	-1.168
	180	-1.999	-.664	.670	-1.335
		-1.999	-.591	.816	-1.407
a= 2	30	-1.996	-.761	.473	-1.234
l= 4	60	-1.983	-1.098	-.212	-.885
	90	-1.955	-1.254	-.553	-.701
	120	-1.910	-1.053	-.196	-.857
	150	-1.906	-.709	.487	-1.197
	180	-1.961	-.588	.785	-1.373
		-2.000	-.665	.668	-1.334
a = .5	30	-2.000	-.833	.334	-1.167
l = 2	60	-1.999	-1.166	-.332	-.833
	90	-1.995	-1.328	-.660	-.667
	120	-1.969	-1.140	-.310	-.829
	150	-1.888	-.752	.383	-1.136

HOOP STRESS OUTSIDE ($\frac{(\sigma_s)_m}{\mu_m \delta}$); $\beta = \infty$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		1.999	.023	.047	1.976
a = 1	30	1.999	.518	.037	1.481
l = 2.25	60	2.002	1.512	.023	.489
	90	2.036	2.039	.041	-.002
	120	2.219	1.681	.143	.537
	150	2.741	1.015	.289	1.726
	180	2.927	.438	-.049	2.488
		2.000	.028	.057	1.971
a = 1	30	2.003	.528	.053	1.474
l = 3	60	2.015	1.532	.050	.482
	90	2.050	2.056	.063	-.006
	120	2.125	1.616	.106	.509
	150	2.161	.639	.116	1.522
	180	2.036	.024	.012	2.011
		2.000	.015	.031	1.984
a = 1	30	2.002	.516	.031	1.485
l = 4	60	2.008	1.520	.033	.487
	90	2.022	2.031	.040	-.008
	120	2.040	1.546	.051	.494
	150	2.031	.537	.043	1.493
	180	2.002	.012	.021	1.990
		2.000	.005	.010	1.994
a = 1	30	2.000	.505	.010	1.494
l = 6	60	2.002	1.507	.012	.495
	90	2.005	2.009	.014	-.004
	120	2.007	1.511	.015	.495
	150	2.003	.508	.012	1.495
	180	2.000	.004	.009	1.995
		2.001	.188	.375	1.813
a = 2	30	2.009	.680	.351	1.329
l = 4	60	2.040	1.671	.302	.368
	90	2.110	2.196	.281	-.085
	120	2.222	1.780	.338	.442
	150	2.234	.810	.386	1.424
	180	2.097	.201	.305	1.895
		1.999	.002	.005	1.997
a = .5	30	1.999	.500	.002	1.498
l = 2	60	2.000	1.501	.001	.499
	90	2.011	2.013	.014	-.001
	120	2.075	1.565	.056	.509
	150	2.277	.700	.122	1.577
	180	2.087	.010	-.066	2.077

HOOP STRESS OUTSIDE $\left(\frac{(\sigma_S)_m}{\mu_m \delta}\right)$; $\beta = 3$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
		2.000	-.008	-.017	2.008
a = 1	30	2.000	.493	-.012	1.506
l = 2.25	60	1.998	1.495	-.007	.503
	90	1.981	1.978	-.023	.002
	120	1.890	1.394	-.100	.495
	150	1.629	.211	-.205	1.417
	180	1.536	-.130	.202	1.666
		1.999	-.013	-.027	2.013
a = 1	30	1.998	.486	-.025	1.512
l = 3	60	1.992	1.483	-.024	.508
	90	1.974	1.970	-.033	.004
	120	1.937	1.439	-.058	.497
	150	1.919	.429	-.060	1.489
	180	1.982	-.006	.005	1.988
		1.999	-.007	-.015	2.007
a = 1	30	1.998	.491	-.015	1.507
l = 4	60	1.995	1.489	-.016	.506
	90	1.988	1.983	-.020	.004
	120	1.979	1.476	-.026	.503
	150	1.984	.481	-.021	1.503
	180	1.998	-.005	-.009	2.003
		1.999	-.002	-.005	2.002
a = 1	30	1.999	.497	-.005	1.502
l = 6	60	1.998	1.496	-.006	.502
	90	1.997	1.995	-.007	.002
	120	1.996	1.494	-.007	.502
	150	1.998	.495	-.006	1.502
	180	1.999	-.002	-.004	2.002
		1.999	-.091	-.181	2.090
a = 2	30	1.995	.411	-.171	1.583
l = 4	60	1.979	1.414	-.150	.565
	90	1.944	1.898	-.148	.046
	120	1.888	1.351	-.185	.537
	150	1.882	.344	-.194	1.538
	180	1.951	-.083	-.118	2.034
		2.000	-.001	-.002	2.001
a = .5	30	2.000	.499	0.000	1.5
l = 2	60	1.999	1.499	0.000	.5
	90	1.994	1.993	-.007	0.0
	120	1.962	1.465	-.030	.496
	180	1.956	.004	.052	1.951

HOOP STRESS OUTSIDE $\left(\frac{(\sigma_S)_m}{\mu_m \delta}\right)$; $\beta = 1/3$

	θ_2	$\delta_1 = \delta_2 = \delta$	$\delta_1 = \delta, \delta_2 = 0$	$\delta_1 = -\delta_2 = \delta$	$\delta_1 = 0, \delta_2 = \delta$
		2.000	-.005	-.011	2.005
a = 1	30	2.000	.495	-.008	1.504
1 = 2.25	60	1.999	1.497	-.005	.502
	90	1.989	1.988	-.012	.001
	120	1.937	1.443	-.050	.494
	150	1.788	.342	-.103	1.445
	180	1.735	-.094	.075	1.830
		1.999	-.007	-.015	2.007
a = 1	30	1.999	.492	-.014	1.507
1 = 3	60	1.995	1.490	-.014	.504
	90	1.985	1.983	-.018	.002
	120	1.964	1.465	-.032	.498
	150	1.953	.459	-.034	1.493
	180	1.989	-.004	0.000	1.994
		1.999	-.004	-.008	2.004
a = 1	30	1.999	.495	-.008	1.504
1 = 4	60	1.997	1.494	-.009	.503
	90	1.993	1.990	-.011	.002
	120	1.988	1.486	-.015	.501
	150	1.990	.489	-.012	1.501
	180	1.999	-.003	-.005	2.002
		1.999	-.001	-.003	2.001
a = 1	30	1.999	.498	-.003	1.501
1 = 6	60	1.999	1.497	-.003	.501
	90	1.998	1.997	-.004	.001
	120	1.997	1.496	-.004	.501
	150	1.998	.497	-.003	1.501
	180	1.999	-.001	-.002	2.001
		1.999	-.052	-.105	2.052
a = 2	30	1.997	.449	-.098	1.548
1 = 4	60	1.988	1.451	-.086	.537
	90	1.968	1.942	-.082	.025
	120	1.936	1.416	-.102	.519
	150	1.933	.411	-.110	1.522
	180	1.972	-.051	-.075	2.023
		2.000	0.000	-.001	2.0
a = .5	30	2.000	.499	0.000	1.5
1 = 2	60	1.999	1.499	0.000	.5
	90	1.996	1.996	-.004	0.0
	120	1.978	1.480	-.016	.497
	150	1.920	.441	-.037	1.479
	180	1.975	0.000	.025	1.974

HOOP STRESS OUTSIDE ($\sigma_3/\mu\delta$); $\theta = 1$

	θ_2	$\delta_1=\delta_2=\delta$	$\delta_1=\delta, \delta_2=0$	$\delta_1=-\delta_2=\delta$	$\delta_1=0, \delta_2=\delta$
		1.999	.009	.018	1.990
a = 1	30	1.999	.506	.014	1.492
1 =2.25	60	2.000	1.504	.008	.495
	90	2.014	2.015	.017	-.001
	120	2.087	1.574	.060	.513
	150	2.296	.709	.122	1.586
	180	2.370	.165	-.040	2.205
		2.000	.011	.022	1.988
a = 1	30	2.001	.511	.021	1.489
1 = 3	60	2.006	1.513	.020	.493
	90	2.020	2.022	.025	-.002
	120	2.050	1.546	.043	.503
	150	2.064	.555	.046	1.508
	180	2.014	.009	.003	2.005
		2.000	.006	.012	1.993
a = 1	30	2.000	.506	.012	1.494
1 = 4	60	2.003	1.508	.013	.495
	90	2.009	2.012	.016	-.003
	120	2.016	1.518	.020	.497
	150	2.012	.515	.017	1.497
	180	2.001	.004	.008	1.996
		2.000	.002	.004	1.997
a = 1	30	2.000	.502	.004	1.497
1 = 6	60	2.001	1.502	.004	.498
	90	2.002	2.003	.005	-.001
	120	2.002	1.504	.006	.498
	150	2.001	.503	.005	1.498
	180	2.000	.001	.003	1.998
		2.000	.075	.149	1.925
a = 2	30	2.003	.572	.140	1.431
1 = 4	60	2.016	1.568	.120	.447
	90	2.044	2.078	.113	-.034
	120	2.089	1.613	.137	.475
	150	2.093	.624	.154	1.469
	180	2.038	.078	.118	1.960
		1.999	.001	.002	1.998
a = .5	30	1.999	.500	0.000	1.499
1 = 2	60	2.000	1.500	0.000	.499
	90	2.004	2.005	.005	0.0
	120	2.030	1.526	.022	.503
	150	2.111	.580	.049	1.530
	180	2.034	.003	-.028	2.031

TABLE 4 : Stresses along the boundary of region 1, in the case of two equal inhomogeneities of unit radius deforming equally (Chapter XII), the distance between their centres being three units; (Plane stress case, Poisson's ratio = 1/3).

NORMAL STRESS $\left(\frac{\sigma_n}{\mu_m S}\right)$

θ	$\beta = .5$ $\sigma_1=\sigma_2=\sigma$	$\beta = .5$ $\sigma_1=-\sigma_2=\sigma$	$\beta = 1$ $\sigma_1=\sigma_2=\sigma$	$\beta = 1$ $\sigma_1=-\sigma_2=\sigma$	$\beta = 2$ $\sigma_1=\sigma_2=\sigma$	$\beta = 2$ $\sigma_1=-\sigma_2=\sigma$
0	-1.676	-.411	-2.499	-.986	-3.156	-1.454
30	-1.499	-.228	-2.026	-.342	-2.448	-.443
60	-1.369	.257	-1.734	.426	-2.027	.558
90	-1.411	.500	-1.840	.662	-2.182	.794
120	-1.483	.245	-1.994	.263	-2.402	.283
150	-1.530	-.254	-2.092	-.429	-2.542	-.562
180	-1.546	-.504	-2.124	-.769	-2.588	-.974

HOOP STRESS INSIDE

	-1.241	.531	-1.500	.763	-1.707	.891
30	-1.446	.301	-1.973	.176	-2.395	.049
60	-1.601	-.232	-2.265	-.526	-2.796	-.758
90	-1.567	-.490	-2.159	-.733	-2.634	-.915
120	-1.499	-.236	-2.005	-.324	-2.411	-.379
150	-1.452	.268	-1.907	.373	-2.270	.472
180	-1.437	.519	-1.875	.714	-2.225	.885

HOOP STRESS OUTSIDE

	2.868	-1.352	.499	-2.458	-1.395	-3.368
30	.417	-1.123	.026	-1.571	-.286	-1.937
90	-.694	1.163	-.159	1.754	.267	2.233
120	.054	.605	-.005	.857	-.053	1.064
150	.539	-.760	.092	-1.101	-.264	-1.370
180	.699	-1.464	.124	-2.091	-.334	-2.588

TANGENTIAL STRESS

	.000	0.000	0.000	0.0	.000	0.000
30	.185	.488	.415	.753	.599	.965
60	.060	.413	.106	.535	.142	.625
90	-.047	-.038	-.120	-.108	-.178	-.175
120	-.068	-.458	-.153	-.659	-.222	-.828
150	-.042	-.444	-.093	-.617	-.134	-.760
180	0.000	0.000	0.000	0.0	.000	0.000

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